Computer-Assisted Proofs of Non-Reachability for Finite-Dimensional Linear Control Systems

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Abstract

It is customary to design a control system in such a way that, whatever the chosen control satisfying the constraints, the system does not enter so-called unsafe regions. This work introduces a general computer-assisted methodology to prove that a given finite-dimensional linear control system with compact constraints avoids a chosen unsafe set at a chosen final time T. Relying on support hyperplanes, we devise a functional such that the property of interest is equivalent to finding a point at which the functional is negative. Actually evaluating the functional first requires time-discretisation. We thus provide explicit, fine discretisation estimates for various types of matrices underlying the control problem. Second, computations lead to roundoff errors, which are dealt with by means of interval arithmetic. The control of both error types then leads to rigorous, computer-assisted proofs of non-reachability of the unsafe set. We illustrate the applicability and flexibility of our method in different contexts featuring various control constraints, unsafe sets, types of matrices and problem dimensions.

Keywords: constrained linear control systems, reachability analysis, computer-assisted proofs, interval arithmetic

AMS classification: 49M29, 49M25, 65G30, 34H05.

1 Introduction

This article is dedicated to the rigorous study of non-reachable states of a constrained controlled linear system. More precisely, we are interested in guaranteeing that, at a given time $T > 0$, the control system cannot enter a prescribed *unsafe region*, whatever the choice of control satisfying the given constraints.

We consider the linear autonomous (time invariant) control system

$$\begin{cases} \dot{y}(t) = Ay(t) + Bu(t), \\ y(0) = y_0, \end{cases} \tag{$\mathcal{S}$}$$

where $y_0 \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$.

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Given $y_0 \in \mathbb{R}^n$, a closed convex set $\mathcal{Y}_f \subset \mathbb{R}^n$, a time horizon $T > 0$ and a compact set $\mathcal{U} \subset \mathbb{R}^m$, we investigate the ($\mathcal{U}$-)constrained reachability problem, i.e., the problem of determining if there exists a control u such that the solution to ($\mathcal{S}$) with control u satisfies $y(T) \in \mathcal{Y}_f$, under the additional constraint that $u(t) \in \mathcal{U}$ for a.e. $t \in (0, T)$. If such a control exists, we shall say that $\mathcal{Y}_f$ *is* $\mathcal{U}$*-reachable from* y_0 *in time* T.

Our aim is to develop a general, flexible and certifiable methodology, resting on numerical computations, to show that $\mathcal{Y}_f$ is not $\mathcal{U}$-reachable from y_0 in time T. Ultimately, the interested user should be able to provide all parameters A, B, T, y_0, $\mathcal{U}$ and $\mathcal{Y}_f$ and, whenever that is the case, be returned the mathematically certified assertion that $\mathcal{Y}_f$ is not $\mathcal{U}$-reachable from y_0 in time T.

1.1 Methodology: non-reachability criterion and certification issues

Support functions. Throughout, finite-dimensional spaces $\mathbb{R}^n$, $\mathbb{R}^m$ will be endowed with the standard Euclidean inner products. If we have $C \subset H$ with H a Hilbert space, σ_C will denote the *support function* of C defined by

$$\forall x \in C, \quad \sigma_C(x) := \sup_{y \in C} \langle x, y \rangle.$$

Non-reachability by separation. By means of separating hyperplanes, we will establish a necessary and sufficient criterion for non-reachability, involving a suitably defined function $J : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, in the following form:

$$(\exists p_f \in \mathbb{R}^n, \quad J(p_f) < 0) \qquad \Longleftrightarrow \qquad \mathcal{Y}_f \text{ is not } \mathcal{U}\text{-reachable from } y_0 \text{ in time } T. \tag{1}$$

The precise definition of J (together with Figure 1 to convey the corresponding intuition) will be given in Section 2.1, and involves the support functions $\sigma_{\mathcal{U}}$ and $\sigma_{\mathcal{Y}_f}$, which we assume to be known explicitly.

The proof of (1) is the object of Proposition 2. In the case where $p_f \in \mathbb{R}^n$ such that $J(p_f) < 0$ is found, we will say that p_f is a *dual certificate* (that $\mathcal{Y}_f$ is not $\mathcal{U}$-reachable from y_0 in time T). One should note that such a dual certificate only proves the non-reachability at time T, and not for all $t \in [0, T]$. Sufficient conditions for the non-reachability at all times $t \in [0, T]$ are proposed in Proposition 7 and the following remark.

Computer-assisted proof of non-reachability. In what follows, we will exploit this criterion by producing vectors that satisfy it numerically. This raises questions pertaining to the error propagation inherent to every numerical method. More precisely:

Certified approach for non-reachability.

- How can one evaluate the functional J, in order to exhibit an element $p_f \in \mathbb{R}^n$ satisfying property (1) numerically?
- How can one then **certify** the numerical result, which implies non-reachability? That is, guarantee that it is not flawed by various numerical approximations?

In order to carry out these two steps, there will in turn be two main difficulties.

(i) We will not have access to J but only to proxies obtained by discretisation, which we generically denote J_{d}. Indeed, the definition of J involves a time-integral, as well as the solution to a linear ODE involving A^* (which amounts to computing the matrix exponentials $t \mapsto e^{tA^*}$). When these are not known explicitly, we will resort to simple time discretisation schemes (implicit Euler, etc) and provide a **bound on the error in terms of discretisation parameters**. One key aspect of our approach is that these bounds must be derived with explicit constants.

(ii) All computations will lead to **round-off errors**, which must be accounted for. To that end, we will use a Matlab/Octave toolbox called INTLAB (INTerval LABoratory) [35]. This code is an interval arithmetic library, entirely written in Matlab. It provides tools for performing numerical computations with arbitrary-precision arithmetic.

All in all, if for a given $p_f \in \mathbb{R}^n$ one lets $E_{\mathrm{d}}(p_f)$ (for the discretisation errors) and $E_{\mathrm{r}}(p_f)$ (for the round-off errors), we will have

$$J(p_f) \in [J_{\mathrm{d}}(p_f) - E_{\mathrm{d}}(p_f) - E_{\mathrm{r}}(p_f), J_{\mathrm{d}}(p_f) + E_{\mathrm{d}}(p_f) + E_{\mathrm{r}}(p_f)]. \tag{2}$$

Hence, we will take advantage of the fact that if

$$J_{\mathrm{d}}(p_f) + E_{\mathrm{d}}(p_f) + E_{\mathrm{r}}(p_f) < 0,$$

then y_f is not $\mathcal{U}$-reachable from y_0 in time T.

Here we stress that the notion of **certification** we are concerned with has to do with the numerical part of our work. The starting point of this work is a theoretical necessary and sufficient condition for non-reachability. For a given system, we can determine whether it is satisfied numerically. Certifying this part then makes this numerical result theoretically sound, thus producing a **computer-assisted proof**. Another key aspect of our methodology is to return a dual certificate p_f that *certifies* the corresponding mathematical statement: consequently, any user with access to its own discretised version of the functional J with corresponding error estimates, can verify the result upon using interval arithmetic.

1.2 State of the art & connections to existing results

The notion of constraint-free controllability of autonomous linear systems dates back to Kalman's seminal works. Its generalisation to infinite-dimensional systems is more recent. For further details on these concepts, we refer to the review books [10, 23]. Since the 70's, but more specifically in recent years, several works have investigated the addition of further constraints, satisfied whether by the control itself, or by the controlled trajectory.

Some of these works are theoretical in nature, with a focus on unbounded constraints. Particular interest has been given to the problems of exact controllability by positive controls due to their physical relevance [6, 12, 15, 24, 28, 29, 33]. Attention was also paid to adding constraints on the controlled trajectory [11, 25, 26]. Unbounded (sparsity) constraints have also been considered [30, 36].

In this article, we focus on the implementation of a method to numerically certify that a set of unsafe states is unreachable at a given time $T > 0$, for compact constraint sets on the control. Our approach is specific to autonomous linear systems. Regarding more general dynamical systems, closely related questions have been addressed in the past: for instance, how to numerically approximate the reachable set at time T, or guarantee that computed trajectories will not meet the given unsafe set at any time $t > 0$.

In finite dimension, several methods have been elaborated to provide approximations of the reachable set (for example, see the recent survey [1]): among others, let us mention the use of Hamilton-Jacobi type equations [9, 27], the design of barrier functions for trajectories to avoid unsafe regions [16, 31], and set propagation [1]. Let us roughly describe each of these approaches.

In [9, 27], a backwards reachable set is characterised as the zero sublevel set of the viscosity solution to a Hamilton-Jacobi type partial differential equation, with important applications to the safety of automated systems. This is formally related to our approach, as we also characterise non-reachability by the existence of negative values for a certain numerical criterion. As we will see, in this paper the convexity of the reachable set and the linearity of the system allow us to exploit this characterisation to produce numerical certificates of non-reachability.

In [16, 31], the authors introduce the notion of barrier functions, appropriately defined from the system dynamics to ensure that trajectories do not enter an unsafe zone. An important element of these methods is that these certificates are valid for all positive times $t > 0$, a very strong property which is not required in other methods, and in particular in this article. Moreover, the computation of barrier certificates for a given system remains a challenging problem, both theoretically and numerically.

Set propagation is a class of methods for computing a guaranteed overapproximation or underapproximation of the reachable set of continuous systems. Starting from the set of initial states, the idea is to iteratively and adequately propagate a sequence of sets according to the system dynamics [13], which are guaranteed to contain, or be contained in, the reachable set. Such an algorithm has been developed in [19, 20] for finite-dimensional compact convex constraints. An important hurdle is then the so-called *wrapping effect*, which is the accumulation of computational errors. The crux of set propagation techniques is to circumvent this difficulty by using appropriate propagation formulae. In this article, the wrapping effect is avoided using duality and considering the solution to a single backward equation.

Separation arguments, as used in this article, already appear in reachability analysis [2, 17–20]. However, an important contribution we make is recasting it in terms of the sign of the function J, in such a way that interval arithmetic can be applied to certify the end result – a feature which seldom appears in the literature.

Using our approach, one can prove that the reachable set is contained in a half-space. Computing several such half-spaces allows for the creation of a bounded convex polytope guaranteed to contain the reachable set, but this can quickly become computationally expensive, especially as the dimension of the problem increases. There exist other ways to over- or under-approximate reachable sets, which rely on geometric properties. In the special case of ellipsoidal constraint sets, we refer to [17, 18]. More generally, for compact convex constraints, the reachable set can be approximated from the outside using support functions [2].

While the above-mentioned works provide theoretical criteria for finite dimensions, the case of infinite dimensions remains largely open.

Extensions and perspectives. We make the assumption that the support functions $\sigma_{\mathcal{U}}$ and $\sigma_{\mathcal{Y}_f}$ are known exactly. If it were not the case, our approach could be extended, provided that one has a procedure to numerically evaluate them, together with a way to control the corresponding error.

The approach we have developed can be adapted *mutatis mutandis* to non-homogeneous non-autonomous linear systems of the form $\dot{x}(t) = A(t)x(t) + B(t)u(t) + v(t)$, for some $v \in L^2(0, T; \mathbb{R}^n)$. The price to pay lies in the error formulae, where the exponential matrix e^{tA} is replaced by the resolvent associated with the function $A(\cdot)$. The resulting formulae would then be slightly less accurate than those we obtained.

Another relevant issue would concern non-reachability in fixed time for non-linear control systems under control sampling, see e.g. [5]. This amounts to imposing specific constraints on the control, assuming it to be piecewise constant with values in a given prescribed set. In the case of a linear system, our approach would apply provided that one provides efficient ways to compute or approximate the support function of these particular types of constraint sets.

In the same vein, the reachability criterion can be extended without effort to Hilbert spaces (see for instance [22] for infinite-dimensional time optimal control problems). This is why we expect our method to accommodate **infinite-dimensional linear control systems**, provided that the space discretisation errors are also estimated. This will be the subject of further work, focusing in particular on the heat equation.

Our work adresses non-reachability. The natural complementary question is that of reachability: can one provide certified methods to show that a target y_f (or more generally, a set $\mathcal{Y}_f$) is reachable? We intend to tackle this problem as well, using similar geometric ideas.

Finally, a more prospective research direction is to investigate generalisations to non-linear control systems. It is obvious that the methodology will have to be thoroughly modified, since our approach fundamentally rests on the linearity of L_T.

Outline of the article. In Section 2, we introduce the criterion J and specify the separation argument, which allows us to recast the non-reachability property. Section 3 focuses on numerical methods for calculating J, using several possible discrete versions. Their relevance is discussed based on the available information about A and its matrix exponential, and in each case, we provide fully explicit error bounds. Finally, the Section 4 is entirely devoted to numerical experiments. After specifying the methodology leading to computer-assisted proofs of non-reachability, we apply it to three examples, with variable dimensions and constraint sets. We present concrete statements, each rigorously proven using our computer-assisted methodology.

2 Non-reachability by separation

2.1 Main result

Consider the linear autonomous control system

$$\begin{cases} \dot{y}(t) = Ay(t) + Bu(t), \quad t \in [0,T], \\ y(0) = y_0, \end{cases} \tag{$\mathcal{S}$}$$

where $y_0 \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, with $u \in E := L^2(0,T;\mathbb{R}^m)$. Recall that we make the following assumption regarding the constraint set $\mathcal{U}$ and the unsafe set $\mathcal{Y}_f$:

$$\mathcal{U} \text{ is compact}, \quad \mathcal{Y}_f \text{ is closed and convex.} \tag{H}$$

The solution to ($\mathcal{S}$) at the final time T is characterised by Duhamel's formula and writes

$$y(T) = e^{TA}y_0 + L_T u, \qquad \text{where} \qquad L_T u := \int_0^T e^{(T-t)A}Bu(t)\,dt.$$

Letting $L(H_1,H_2)$ denote the set of linear continuous operators between two Hilbert spaces H_1 and H_2, it is standard that L_T defines an operator in $L(E,\mathbb{R}^n)$. Its adjoint, $L_T^* \in L(\mathbb{R}^n,E)$, is defined for $p_f \in \mathbb{R}^n$ by $L_T^* p_f(t) = B^* p(t)$, where p solves the backward adjoint equation.

$$\begin{cases} \dot{p}(t) + A^* p(t) = 0, \quad t \in [0,T] \\ p(T) = p_f, \end{cases} \tag{3}$$

As already mentioned, the key aspect of our approach hinges on the assertion (1), where J denotes the so-called *dual* functional, defined by

$$\forall p_f \in \mathbb{R}^n, \qquad J(p_f) := \int_0^T \sigma_{\mathcal{U}}(L_T^* p_f(t))\,\mathrm{d}t + \sigma_{\mathcal{Y}_f}(-p_f) + \langle y_0, e^{TA^*}p_f\rangle. \tag{4}$$

Remark 1. *When $\mathcal{U}$ is convex, the functional J can be understood as a dual functional associated to a primal problem, in the sense of Fenchel-Rockafellar. More details are provided in Appendix A. This interpretation leads us to consider useful algorithms that perform a descent over J in order to find dual certificates, as explained in Section 4.*

The following result describes the crucial argument underpinning our method, which is illustrated by Figure 1.

Proposition 2. *Assume that (H) holds. Then, there exists $p_f \in \mathbb{R}^n$ such that $J(p_f) < 0$ if and only if $\mathcal{Y}_f$ is not $\mathcal{U}$-reachable from y_0 in time T.*

Proof. Let $E_{\mathcal{U}} := \{u \in E,\ u(t) \in \mathcal{U} \text{ for a.e. } t \in (0,T)\}$. With this notation in place, $\mathcal{Y}_f$ is not $\mathcal{U}$-reachable from y_0 in time T if and only if the set $(\mathcal{Y}_f - e^{TA}y_0) \cap L_T E_{\mathcal{U}}$ is empty, where

$$\mathcal{Y}_f - e^{TA}y_0 = \{y - e^{TA}y_0, y \in Y_f\}.$$

Using the basic relation $\sigma_{C-\{y\}}(z) = \sigma_C(z) - \langle y, z\rangle$, we have

$$\sigma_{\mathcal{Y}_f - e^{TA}y_0}(-p_f) = \sigma_{\mathcal{Y}_f}(-p_f) + \langle e^{TA}y_0, p_f\rangle = \sigma_{\mathcal{Y}_f}(-p_f) + \langle y_0, e^{TA^*}p_f\rangle$$

As a result, the function J defined in (4) rewrites

$$J(p_f) = \int_0^T \sigma_{\mathcal{U}}(L_T^* p_f(t))\,\mathrm{d}t + \sigma_{\mathcal{Y}_f - e^{TA}y_0}(-p_f) = \sigma_{E_{\mathcal{U}}}(L_T^* p_f) + \sigma_{\mathcal{Y}_f - e^{TA}y_0}(-p_f),$$

where the interchange of integration and supremum is justified, see e.g. [34, Theorem 14.60].

Now assume that we have found p_f such that $J(p_f) < 0$. Then

$$\sigma_{E_{\mathcal{U}}}(L_T^* p_f) = \sup_{u \in E_{\mathcal{U}}} \langle u, L_T^* p_f\rangle = \sup_{u \in E_{\mathcal{U}}} \langle L_T u, p_f\rangle < -\sigma_{\mathcal{Y}_f - e^{TA}y_0}(-p_f) = \inf_{y_f \in \mathcal{Y}_f} \langle y_f - e^{TA}y_0, p_f\rangle,$$

showing that one cannot find $u \in E_{\mathcal{U}}$ and $y_f \in \mathcal{Y}_f$ such that $L_T u = y_f - e^{TA}y_0$ and hence that $\mathcal{Y}_f$ is not $\mathcal{U}$-reachable from y_0 in time $T > 0$.

Conversely, suppose that $\mathcal{Y}_f$ is not $\mathcal{U}$-reachable from y_0 in time T. Then, since $\mathcal{U}$ is compact, it follows from a Lyapunov argument that the set of reachable states (from 0 in time T), i.e., the set $L_T E_{\mathcal{U}}$, is compact and convex (see e.g. [21, Theorem 1A, Theorem 3 and Lemma 4A in Section 2.2]). The set $\mathcal{Y}_f - e^{TA}y_0$ is closed and convex.

By assumption, these two sets do not intersect, hence we may strictly separate them: there exists $p_f \in \mathbb{R}^n \setminus \{0\}$ such that

$$\sigma_{E_{\mathcal{U}}}(L_T^* p_f) = \sup_{w \in L_T E_{\mathcal{U}}} \langle w, p_f\rangle < \inf_{y_f \in \mathcal{Y}_f} \langle y_f - e^{TA}y_0, p_f\rangle = -\sigma_{\mathcal{Y}_f - e^{TA}y_0}(-p_f)$$

which amounts to $J(p_f) < 0$. □

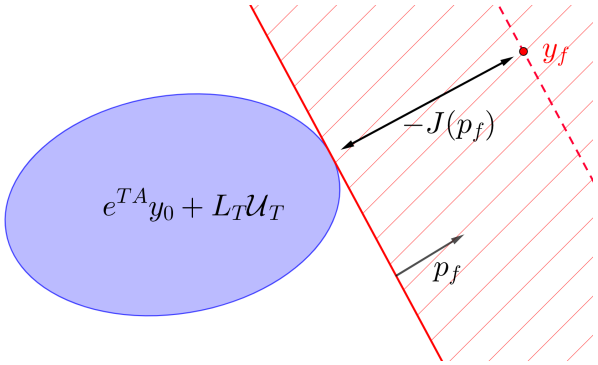

Figure 1: Reachable set $e^{TA}y_0 + L_T E_{\mathcal{U}}$, hyperplane associated to the dual certificate p_f, and corresponding scalar $J(p_f)$ given by (6), for a singleton $\mathcal{Y}_f = \{y_f\}$.

Remark 3. *By positive 1-homogeneity of support functions, J is also positively 1-homogeneous, meaning that $J(\lambda p_f) = \lambda J(p_f)$ for all $\lambda \geq 0$, $p_f \in \mathbb{R}^n$. In particular, if there exists p_f such that $J(p_f) < 0$, then $\inf_{p_f \in \mathbb{R}^n} J(p_f) = -\infty$.*

Remark 4. *We could also consider proving that $\mathcal{Y}_f$ is not $\mathcal{U}$-reachable from a full set of initial states $\mathcal{Y}_0 \subset \mathbb{R}^n$ in time T, in which case, defining*

$$\forall p_f \in \mathbb{R}^n, \qquad J(p_f) = \int_0^T \sigma_{\mathcal{U}}(L_T^* p_f(t))\,\mathrm{d}t + \sigma_{\mathcal{Y}_f}(-p_f) + \sigma_{\mathcal{Y}_0}(e^{TA^*}p_f)$$
$$= \int_0^T \sigma_{\mathcal{U}}(L_T^* p_f(t))\,\mathrm{d}t + \sigma_{\mathcal{Y}_f - e^{TA}\mathcal{Y}_0}(-p_f),$$

the result of Proposition 2 holds as is under the assumption that the set $\mathcal{Y}_f - e^{TA}\mathcal{Y}_0$ is closed and convex: this is the case for instance if $\mathcal{Y}_f$ is closed and convex, and $\mathcal{Y}_0$ convex and compact.

Remark 5. *The above proposition gives a necessary and sufficient condition for non-reachability. It is worth pointing out that, without any assumptions on the sets $\mathcal{U}$, $\mathcal{Y}_0$, $\mathcal{Y}_f$, the above criterion remains a sufficient condition for non-reachability, as it yields a strict separating hyperplane between $\mathcal{Y}_f$ and $e^{TA}\mathcal{Y}_0 + L_T E_{\mathcal{U}}$. In that case however, situations where these sets are disjoint but not separable by a hyperplane (typically if $\mathcal{Y}_f$ is not convex) are then undetectable by our approach.*

Remark 6. *As mentioned in the introduction, the above can be linked (at least formally) to the Hamilton-Jacobi characterisation of some reachable sets [9, 27]. Indeed, formally, in optimal control problems, the value function is solution to a Hamilton Jacobi type equation. Now, for our control problem, the value function writes*

$$S(y_f) := \begin{cases} 0 & \text{if } y_f \text{ is reachable,} \\ +\infty & \text{otherwise,} \end{cases}$$

so we see that the non-reachable set is characterised as the strict zero superlevel set $\{y,\ S(y) > 0\}$ of S. Note that S is a very singular function, and its numerical computation is not tractable, whereas a geometrical approach using support functions leads to a convex function on which a descent algorithm is then implemented, which is much more amenable and prone to numerical certification.

2.2 Unsafe sets and minimal times

As previously mentioned, we assume throughout that we know an explicit formula for both functions $\sigma_{\mathcal{U}}$ and $\sigma_{\mathcal{Y}_f}$, which will be the case in the range of examples we will provide. For instance, for $\mathcal{U}$ defined by the most standard box constraints $\ell_i \leq u_i \leq L_i$ for $i \in \{1, \dots, m\}$, one has with $\ell = (\ell_i)$, $L = (L_i)$ the explicit formula

$$\forall u \in \mathbb{R}^m, \quad \sigma_{\mathcal{U}}(u) = \langle L, u_+ \rangle + \langle \ell, u_- \rangle, \tag{5}$$

where $u_+ = \max(u, 0)$ and $u_- = \min(u, 0)$ refer to the (componentwise) positive and negative parts of u respectively, and multiplications are to be understood componentwise.

Let us now discuss expressions for the functional (4) for some specific, yet natural, choices of sets $\mathcal{Y}_f$.

Chosen unsafe sets $\mathcal{Y}_f$. Most of our examples in this article will be based on, but not limited to, the singleton case $\mathcal{Y}_f = \{y_f\}$. Below, we compute the corresponding functional and explain how one then infers results for a closed ball around y_f, *i.e.*, $\mathcal{Y}_f = \overline{B}(y_f, \varepsilon)$, and even a full half-space associated with y_f. Section 4.3 features a more involved (unbounded) example where $\mathcal{Y}_f$ is a cylinder in $\mathbb{R}^4$, pertaining to the Space rendezvous problem.

Singleton. In the case $\mathcal{Y}_f = \{y_f\}$, one computes $\sigma_{\mathcal{Y}_f}(-p_f) = -\langle y_f, p_f \rangle$, which leads to the functional

$$J(p_f) = \int_0^T \sigma_{\mathcal{U}}(L_T^* p_f(t))\,\mathrm{d}t - \langle y_f, p_f \rangle + \langle y_0, e^{TA^*} p_f \rangle. \tag{6}$$

Ball. In the case of a ball $\mathcal{Y}_f = \overline{B}(y_f, \varepsilon)$ (which recovers the above case with $\varepsilon = 0$), we find

$$\sigma_{\mathcal{Y}_f}(-p_f) = -\langle y_f, p_f\rangle + \varepsilon\|p_f\|,$$

hence we uncover the same functional up to the additional term $\varepsilon\|p_f\|$.

In practice, this has the following implication: given y_f, assume that we have found p_f such that $J(p_f) < 0$ with J given by (6). Then $\overline{B}(y_f, \varepsilon)$ is not $\mathcal{U}$-reachable from y_0 in time $T > 0$ for any $\varepsilon < -J(\frac{p_f}{\|p_f\|})$. Hence, once a target y_f is fixed, we will only be concerned with the functional J given by (6). If p_f is found such that $J(p_f) < 0$, we thus obtain a full ball around y_f that is not $\mathcal{U}$-reachable from y_0 in time $T > 0$.

Half-space. We now show how an unreachable half-space can be constructed from any target y_f. For the sake of this remark, when considering the associated functional (6), we highlight the dependence of J on the target y_f, by writing $J(p_f; y_f)$ instead of just $J(p_f)$.

Now, assume that $\alpha := J(p_f; y_f)$ has been computed for a given $p_f \in \mathbb{R}^n$. For any $\tilde{y_f} \in \mathbb{R}^n$, we have the relation

$$J(p_f; \tilde{y_f}) = J(p_f; y_f) + \langle y_f - \tilde{y_f}, p_f\rangle.$$

Hence, Proposition 1 shows that, independently of the sign of α, any vector in the half-space

$$\{\tilde{y_f} \in \mathbb{R}^n,\ \langle \tilde{y_f} - y_f, p_f\rangle > \alpha\},$$

is not $\mathcal{U}$-reachable from y_0 in time T. In other words, calculating $J(p_f; y_f)$ for any p_f immediately provides a full half-space that is not $\mathcal{U}$-reachable from y_0 in time T.

Minimal times. It is interesting to notice that, still in the case where $\mathcal{Y}_f = \{y_f\}$ and assuming we have either $y_0 = 0$ or $y_f = 0$, we can also derive a lower bound on the minimal reachability time. We will exploit this result in obtaining (lower) estimates for minimal times in the case of two control systems in Section 4.

Proposition 7. *Assume that $\mathcal{U} \cap \mathrm{Ker}(B) \neq \emptyset$, and suppose either $y_0 = 0$ or $y_f = 0$.*

If y_f is not $\mathcal{U}$-reachable from y_0 in time T, then it is not reachable for any $\tilde{T} \leq T$ either. Consequently, denoting

$$T^\star(y_0, y_f, \mathcal{U}) = \inf\left\{T > 0,\ y_f \text{ is } \mathcal{U}\text{-reachable from } y_0 \text{ in time } T\right\} \in [0 + \infty],$$

we have $T^\star(y_0, y_f, \mathcal{U}) \geq T$.

Proof. This proposition is standard and its proof is elementary. Let us provide the main argument in the case where $y_0 = 0$ for the sake of completeness. Assume that y_f is $\mathcal{U}$-reachable from 0 in time $\tilde{T}$ by a control $\tilde{u}$. Let $T > \tilde{T}$. Let $v \in \mathcal{U} \cap \mathrm{Ker}B$. Then, the control u defined by $u(t) = v$ for $t \in (0, T - \tilde{T})$ and $u(t) = \tilde{u}(t - T + \tilde{T})$ steers the system from 0 to y_f in time T and satisfies the constraint, hence the conclusion. The end of the proof is straightforward. □

Remark 8. *In the context of proving that a given unsafe set $\mathcal{Y}_f$ is not reachable for all times $t \in [0, T]$, let us note the following equivalence: $\mathcal{Y}_f$ is not $\mathcal{U}$-reachable from y_0 in all times $t \in [0, T]$ if and only if for all $t \in [0, T]$, $\mathcal{Y}_f - e^{tA}y_0$ is not $\mathcal{U}$-reachable from 0 in time t. This leads to a sufficient condition: if $\mathcal{Y}_f - \cup_{t\in[0,T]}\{e^{tA}y_0\}$ is not $\mathcal{U}$-reachable from 0 in time T, then $\mathcal{Y}_f$ is not $\mathcal{U}$-reachable from y_0 in all times $t \in [0, T]$.*

3 Discretisation and error estimates

This section presents several discretisations and corresponding error estimates for the dual functional (4). Error estimates are given using standard Hermitian norms (over $\mathbb{C}^n$ and $\mathbb{C}^m$), always denoted by $\|\cdot\|$. The same notation $\|\cdot\|$ will also be used for the corresponding operator norms, that of matrices in $\mathbb{C}^{n\times n}$, $\mathbb{C}^{m\times n}$ and $\mathbb{C}^{n\times m}$.

As discussed in the introduction, we make the reasonable assumption that we have access to an explicit formula for $\sigma_{\mathcal{U}}$ (and $\sigma_{\mathcal{Y}_f}$). Also recall that $\mathcal{U}$ is compact, and M denotes a positive constant such that $\|v\| \leq M$ for all $v \in \mathcal{U}$. In particular, it is easily seen that

$$\forall x, y \in \mathbb{R}^m, \quad \sigma_{\mathcal{U}}(x) \leq \sigma_{\mathcal{U}}(x-y) + \sigma_{\mathcal{U}}(y) \leq M\|x-y\| + \sigma_{\mathcal{U}}(y),$$

which implies that $\sigma_{\mathcal{U}}$ is M-Lipschitz.

3.1 Partial discretisation for a known adjoint exponential

To evaluate the dual functional (4) at a given point p_f, one must compute a time integral, and solve the backward equation (3). Given that $\sigma_{\mathcal{U}}$ will generally not be better behaved than Lipschitz, we will stick to time-discretisation schemes that are of order 1, whether for computing integrals or for integrating ODEs.

Even when one has access to an explicit solution for the backward equation (3), the integral will seldom be computable (or at the cost of cumbersome computations). This is why we first consider the case of discretising the integral but not the backward equation (3).

We define

$$N_t \in \mathbb{N}^*, \quad \Delta t = \frac{T}{N_t}, \quad t_k = k\Delta t \text{ for } k \in \{0, \dots, N_t\}.$$

For a fixed $p_f \in \mathbb{R}^n$, we let $t \mapsto p(t)$ be the solution to (3), i.e., $p(t) = e^{(T-t)A^*}p_f$, and consider $J_{\mathrm{d},1}$, the first discretised version of J given by

$$J_{\mathrm{d},1}(p_f) := \Delta t \sum_{k=1}^{N_t} \sigma_{\mathcal{U}}(B^* p(t_k)) + \sigma_{\mathcal{Y}_f}(-p_f) + \langle y_0, p(0)\rangle. \tag{7}$$

Proposition 9. *For a given $p_f \in \mathbb{R}^n$, it holds that*

$$|J(p_f) - J_{\mathrm{d},1}(p_f)| \leq \frac{1}{2}\Delta t \; MT\|B\|\Big(\sup_{t\in[0,T]} \|e^{tA^*}\|\Big)\|A^* p_f\|.$$

Proof. Recall that $\sigma_{\mathcal{U}}$ is M-Lipschitz continuous; therefore, we have for all $s, t \in [0,T]$

$$\big|\sigma_{\mathcal{U}}\big(L_T^* p_f(s)\big) - \sigma_{\mathcal{U}}\big(L_T^* p_f(t)\big)\big| \leq M\|B^* p(s) - B^* p(t)\| \leq M\|B\|\|p(t) - p(s)\|.$$

We can now establish the bound

$$\big|\sigma_{\mathcal{U}}\big(L_T^* p_f(s)\big) - \sigma_{\mathcal{U}}\big(L_T^* p_f(t)\big)\big| \leq M\|B\| \sup_{t\in[0,T]} \|A^* p(t)\| \, |t-s|.$$

We have proved that $t \mapsto \sigma_{\mathcal{U}}\big(L_T^* p_f(t)\big)$ is Lipschitz continuous. Recalling the standard estimate

$$\left|\int_0^T f(t)\,\mathrm{d}t - \Delta t \sum_{k=1}^{N_t} f(t_k)\right| \leq \frac{1}{2}KT\,\Delta t$$

for a K-Lipschitz function $f : [0,T] \to \mathbb{R}$, we end up with

$$\left|\int_0^T \sigma_{\mathcal{U}}(B^* p(t))\,\mathrm{d}t - \Delta t \sum_{k=1}^{N_t} \sigma_{\mathcal{U}}(B^* p(t_k))\right| \leq \frac{1}{2}\Delta t MT\|B\| \sup_{t\in[0,T]} \|A^* p(t)\|,$$

thus, the previously announced estimate readily follows, using the definition of $p(t)$:

$$\begin{aligned}\sup_{t\in[0,T]}\|A^*p(t)\| &= \sup_{t\in[0,T]}\|A^*e^{(T-t)A^*}p_f\| = \sup_{t\in[0,T]}\|A^*e^{tA^*}p_f\| \\ &= \sup_{t\in[0,T]}\|e^{tA^*}A^*p_f\| \leq \sup_{t\in[0,T]}\|e^{tA^*}\|\|A^*p_f\|.\end{aligned}$$

□

Jordan-Chevalley decomposition. Even if one knows the matrix exponentials $t \mapsto e^{tA^*}$ (or equivalently the matrix exponentials $t \mapsto e^{tA}$), it is still necessary to provide an upper bound for $\sup_{t\in[0,T]} \| e^{tA^*}\| = \sup_{t\in[0,T]} \|e^{tA}\|$ to make the bound in Proposition 9 useful.

Assume that we have access to the Jordan-Chevalley decomposition of A in the following sense: we have $A = D + N$ where D is diagonalisable, N is nilpotent with index ℓ, the two matrices D and N commute. Then, of course, e^{tA} is obtained by

$$\forall t \in \mathbb{R}, \qquad e^{tA} = e^{tD}\sum_{k=0}^{\ell-1}\frac{N^k}{k!}t^k = e^{tD}Q_\ell(tN), \tag{8}$$

where Q_ℓ is the polynomial $x \mapsto \sum_{k=0}^{\ell-1}\frac{x^k}{k!}$. Assume further that we have access to the transition matrix P that diagonalises D, i.e., $\mathrm{diag}(\Lambda) = P^{-1}DP$ where $\Lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ stores the eigenvalues of A.

Consequently, we have

$$e^{tA} = Pe^{t\Lambda}P^{-1}Q_\ell(tN),$$

which leads to the estimate

$$\sup_{t\in[0,T]}\|e^{tA}\| \leq \kappa(P)e^{\mu T}Q_\ell(\|N\|T),$$

where $\mu := \max(\{\mathrm{Re}(\lambda_i),\ i \in \{0, \ldots, n\})$ is the spectral abscissa of A, and $\kappa(P) = \|P\|\|P^{-1}\|$ stands for the condition number of the transition matrix P.

From these estimates, we derive the error formula below, in the case where the Jordan-Chevalley decomposition is known.

Corollary 10. *Let us assume that we know the explicit Jordan-Chevalley decomposition of A, in the form $A = D + N$. Then for a given $p_f \in \mathbb{R}^n$, there holds*

$$|J(p_f) - J_{\mathrm{d},1}(p_f)| \leq \frac{1}{2}\Delta t\ MT\|B\|\|A^*p_f\|\kappa(P)e^{\mu T}Q_\ell(\|N\|T).$$

3.2 Full discretisation

We now address the scenario where the adjoint exponential $t \mapsto e^{tA^*}$ is unknown, necessitating the discretisation of the backward equation (3) as well. Assume that a discretisation scheme has been applied that produces $p_k \in \mathbb{R}^n$ for $k \in \{0, \ldots, N_t\}$.

In the next subsection, we will specialise to the Euler implicit scheme for the class of negative semi-definite matrices.

The fully discretised version of J then reads

$$J_{\mathrm{d},2}(p_f) := \Delta t\sum_{k=1}^{N_t}\sigma_{\mathcal{U}}(B^*p_k) + \sigma_{\mathcal{Y}_f}(-p_f) + \langle y_0, p_0\rangle. \tag{9}$$

Proposition 11. *For a given $p_f \in \mathbb{R}^n$ and vectors $p_k \in \mathbb{R}^n$, $k \in \{0, \dots, N_t\}$, there holds*

$$|J(p_f) - J_{\mathrm{d},2}(p_f)| \leq \Delta t \; M\|B\| \left(\frac{1}{2} T \|A^* p_f\| \sup_{t\in[0,T]} \|e^{tA^*}\| + \sum_{k=1}^{N_t} \|p(t_k) - p_k\| \right) + \|y_0\| \|p(0) - p_0\|.$$

The proof is straightforward and left to the reader, as it primarily involves providing an estimate for $|J_{\mathrm{d},1}(p_f) - J_{\mathrm{d},2}(p_f)|$ and combining it with the estimate given from Proposition 9.

We now explore the application of the simplest possible scheme, that is the Euler explicit scheme:

$$\begin{cases} p_{N_t} = p_f \\ p_k = (\mathrm{Id} + \Delta t A^*) p_{k+1} \quad \forall k \in \{0, \dots, N_t - 1\}. \end{cases} \tag{10}$$

Note that the Euler implicit scheme could also be employed and would yield similar results. It is then standard (see e.g. [32, Section 11.3.2]) that

$$\forall k \in \{0, \dots, N_t\}, \quad \|p(t_k) - p_k\| \leq \frac{1}{2} \Delta t \; (T - t_k) \Big(\sup_{t\in[t_k,T]} \|p''(t)\| \Big) e^{\|A\|T}.$$

Given that $p''(t) = e^{(T-t)A^*} (A^*)^2 p_f$, this leads to the estimate

$$\begin{aligned} \forall k \in \{0, \dots, N_t\}, \quad \|p(t_k) - p_k\| &\leq \frac{1}{2} \Delta t \; (T - t_k) \Big(\sup_{t\in[t_k,T]} \|e^{tA^*}\| \Big) e^{\|A\|T} \|(A^*)^2 p_f\| \\ &\leq \frac{1}{2} \Delta t \; (T - t_k) e^{2\|A\|T} \|(A^*)^2 p_f\| \end{aligned}$$

We acknowledge that constants appearing in the above might slightly be improved. All in all, we thus find following the global estimate.

Proposition 12. *For a given $p_f \in \mathbb{R}^n$ and vectors $p_k \in \mathbb{R}^n$, $k \in \{0, \dots, N_t\}$ defined according to the Euler explicit scheme (10), it holds that*

$$|J(p_f) - J_{\mathrm{d},2}(p_f)| \leq \frac{1}{2} \Delta t \; T \left[M\|B\| \left(e^{\|A\|T} \|A^* p_f\| + \frac{1}{2} T e^{2\|A\|T} \|(A^*)^2 p_f\| \right) + \|y_0\| e^{2\|A\|T} \|(A^*)^2 p_f\| \right].$$

Proof. The only step that requires detailed explanation is the estimation of the sum of the errors $\|p(t_k) - p_k\|$, obtained by writing

$$\sum_{k=1}^{N_t} \|p(t_k) - p_k\| \leq \frac{1}{2} \Delta t \, e^{2\|A\|T} \|(A^*)^2 p_f\| \sum_{k=1}^{N_t} (T - t_k) = \frac{1}{2} \Delta t \, e^{2\|A\|T} \|(A^*)^2 p_f\| \frac{T}{N_t} \sum_{k=1}^{N_t} (N_t - k).$$

The sum $\sum_{k=1}^{N_t} (N_t - k)$ equals $\frac{(N_t-1)N_t}{2}$; therefore

$$\sum_{k=1}^{N_t} \|p(t_k) - p_k\| = \frac{1}{4} \Delta t \, e^{2\|A\|T} \|(A^*)^2 p_f\| T(N_t - 1) \leq \frac{1}{4} T^2 e^{2\|A\|T} \|(A^*)^2 p_f\|.$$

□

This estimate has one major drawback: it diverges exponentially fast as a function of T, making the investigation of non-$\mathcal{U}$-reachability challenging, even for moderate times $T > 0$, especially if the matrix norm $\|A\|$ is large.

3.3 Full discretisation for a symmetric negative semidefinite matrix

The purpose of this subsection is to exhibit a class of matrices, that of symmetric negative semidefinite matrices, for which refined estimates can be derived without the errors exponentially diverging errors as a function of time T.

Even though such matrices are diagonalisable, computing their exponential can become intractable for large sizes, so that one needs to resort to discretisation for the backward equation (3). The implicit Euler scheme below is well suited to that situation:

$$\begin{cases} p_{N_t} = p_f \\ (\mathrm{Id} - \Delta t A^*) p_k = p_{k+1} \quad \forall k \in \{0, \dots, N_t - 1\}. \end{cases} \tag{11}$$

It always makes sense provided Δt is small enough, and in the case where the matrix A is a negative semidefinite symmetric matrix, the Euler implicit scheme is well-defined whatever the value of $\Delta t > 0$.

Assume we are given a symmetric positive semidefinite matrix C, diagonalised in the form $C = PDP^{-1}$, with D diagonal and P a orthogonal transition matrix, we may define $\varphi(C)$ for any function $\varphi : [0, +\infty) \to \mathbb{R}$ by $\varphi(C) = P\varphi(D)P^{-1}$ with componentwise application of φ on the diagonal. This definition obviously agrees with the usual matrix exponential and rational fractions whose poles avoid $[0, +\infty)$.[1] Using that $\kappa(P) = 1$, one has for all such functions

$$\|\varphi(C)\| = \|\varphi(D)\| \leq \sup_{x \geq 0} |\varphi(x)|, \tag{12}$$

Proposition 13. *Assume that A is a negative semidefinite symmetric matrix, and let $p_f \in \mathbb{R}^n$. Then the error between the solution to the backward ODE (3) and its implicit Euler discretisation (11) satisfies*

$$\forall k \in \{0, \dots, N_t\}, \quad \|p(t_k) - p_k\| \leq \frac{1}{2} \Delta t \, \|A^* p_f\|. \tag{13}$$

Proof. By definition, for all $k \in \{0, \dots, N_t\}$, we have

$$p(t_k) - p_k = \left[e^{(T - t_k)A^*} - (\mathrm{Id} - \Delta t A^*)^{-(N_t - k)} \right] p_f.$$

Hence me may write

$$p(t_k) - p_k = -\Delta t \, \varphi_{N_t - k}(-\Delta t A^*) A^* p_f,$$

where for $k \in \mathbb{N}^*$, the function φ_k is defined for $x > 0$ by

$$\varphi_k(x) := \frac{e^{-kx} - (1 + x)^{-k}}{x},$$

extended by continuity at $x = 0$ by $\varphi_k(0) := 0$.

Estimating, we find

$$\|p(t_k) - p_k\| \leq \Delta t \, \|\varphi_{N_t - k}(-\Delta t A^*)\| \, \|A^* p_f\| \leq \Delta t \sup_{x \geq 0} |\varphi_{N_t - k}(x)| \|A^* p_f\|$$

Let us conclude by proving that $\sup_{x \geq 0} |\varphi_k(x)| \leq \frac{1}{2}$ for all $k \geq 1$. First, a routine study shows that the function $x \mapsto e^{-x}(1 + x) - 1 + \frac{1}{2}x^2$ is nonnegative for all $x \geq 0$, so that

$$|\varphi_1(x)| = \frac{1}{x} \left[\frac{1}{1 + x} - e^{-x} \right] \leq \frac{1}{2} x, \tag{14}$$

[1] There are of course much more general definitions for functions of matrices [14], but in the present setting this definition will suffice.

which combined with the basic estimate $|\varphi_1(x)| \leq \frac{1}{x}\frac{1}{1+x}$ for $x > 0$ yields $|\varphi_1(x)| \leq \frac{1}{2}$ by considering the two cases $x \leq 1$ and $x > 1$. Now for $k \geq 2$, and $x > 0$, we write

$$|\varphi_k(x)| = \frac{1}{x}\left[\frac{1}{1+x}\right]\sum_{j=0}^{k-1} e^{-jx}\Big(\frac{1}{1+x}\Big)^{k-j-1} = |\varphi_1(x)|\sum_{j=0}^{k-1} e^{-jx}\Big(\frac{1}{1+x}\Big)^{k-j-1} \leq |\varphi_1(x)|\frac{k}{(1+x)^{k-1}}.$$

thanks to the bound $e^{-x} \leq \frac{1}{1+x}$. Let us focus on the case $k = 2$. If $x \leq 1$, we have $|\varphi_2(x)| \leq \frac{1}{2}x\frac{2}{1+x} \leq \frac{1}{2}$, and for $x > 1$, $|\varphi_2(x)| \leq \frac{1}{x(1+x)}\frac{2}{1+x} \leq \frac{1}{2}$, hence the result for $k = 2$.

Now for any $k \geq 3$, using the estimate (14), we obtain the inequality

$$|\varphi_k(x)| \leq \frac{kx}{2(1+x)^{k-1}}$$

The right-hand side is maximised at $x = \frac{1}{k-2}$, hence

$$|\varphi_k(x)| \leq \frac{k}{2(k-2)}\Big(\frac{k-2}{k-1}\Big)^{k-1} = \frac{1}{2}\frac{k(k-2)}{(k-1)^2}\Big(\frac{k-2}{k-1}\Big)^{k-3} \leq \frac{1}{2}.$$

□

This entails the following compact estimate for the dual functional.

Proposition 14. *Assume that A is a symmetric negative semidefinite matrix. For a given $p_f \in \mathbb{R}^n$ and vectors $p_k \in \mathbb{R}^n$, $k \in \{0, \dots, N_t\}$ defined according to the Euler implicit scheme (11), there holds*

$$|J(p_f) - J_{\mathrm{d},2}(p_f)| \leq \Delta t\, \|A^* p_f\| \left(TM\|B\| + \frac{1}{2}\|y_0\|\right). \tag{15}$$

Proof. We simply build upon the general estimate of Proposition 11. First, since $-A^*$ is a symmetric positive semidefinite matrix, (12) provides

$$\|e^{tA^*}\| \leq \sup_{x\geq 0} |e^{-tx}| = 1$$

for all $t \geq 0$, and the previous estimate from Proposition 13 for the Euler implicit scheme shows that

$$\sum_{k=1}^{N_t} \|p(t_k) - p_k\| \leq \frac{1}{2}\Delta t\|A^* p_f\| N_t = \frac{1}{2}T\|A^* p_f\|.$$

Remark 15. *We note that similar estimates, not exponentially diverging with T, could also be derived for the broader class of dissipative matrices (i.e., matrices A satisfying $\langle Ax, x\rangle \leq 0$ for all $x \in \mathbb{R}^n$).*

□

4 Numerical approach and examples

In this section, we will illustrate the potential of the approach described in the previous section to study the (non)-reachability of certain targets, in a variety of examples. We present three main example families, respectively related to:

- the control of a streetcar that we wish to control in order to reach a final state in minimal time. This is a well-known toy problem in optimal control theory. We use it to validate our results since the reachable set and minimal times $\big($from $(0,0)^T\big)$ have known explicit formulae;

- the spatial rendezvous problem. We aim at reaching (or avoiding) a given target, corresponding to a space station, for instance the ISS, in a referential centered in the initial position of the spacecraft. We use a dynamic space mechanics model and provide certified lower-bounds on the minimal time needed to reach the target. We then develop a method to prove that a motionless obstacle – e.g. an asteroid – cannot be collided within a predetermined time interval.
- a more academic setting, based on randomly generated negative semi-definite (Jacobi) matrices A (of possibly large dimension). This is designed to investigate cases where computing exponentials becomes out of reach, as well as to explore the effect increasing dimension has on our technique.

Most cases feature a set of the form $\mathcal{Y}_f = \{y_f\}$, hence the function of interest is (6). As explained in Subsection 2, the use of the corresponding functional also allows us to certify that balls around y_f or even half-spaces cannot be reached. The types of constraint sets $\mathcal{U}$ also vary across examples.

4.1 Numerical approach and methodology

In order to numerically verify the non-$\mathcal{U}$-reachability of a given target y_f from y_0 in time T, one must proceed through the following three steps:

1. First, one must compute a discretisation J_d of the functional J, for example $J_{\mathrm{d},1}$ or $J_{\mathrm{d},2}$, with the associated bounds on discretisation errors
2. Then, one must minimise said discretisation in order to find an element p_f such that $J_\mathrm{d}(p_f) < 0$.
3. Finally, one must compute $e(p_f)$ such that $J_\mathrm{d}(p_f) - e(p_f) \leq J(p_f) \leq J_\mathrm{d}(p_f) + e(p_f)$. This is done here using the INTLAB toolbox [35], which, using interval arithmetic, takes into account the rounding errors and added discretisation errors. This leads to the verification that indeed, $J(p_f) \leq J_\mathrm{d}(p_f) + e(p_f) < 0$. If that is not the case, either y_f is reachable, or a finer discretisation or minimisation is required to prove its non-reachability.

Since INTLAB allows for most of usual computation techniques, the second and third steps could be joined. However, interval arithmetic is computationally expensive, hence we first minimise the discretised functional J_d to find p such that $J_\mathrm{d}(p_f) < -\eta$, where η is the typical size of errors $e(p_f)$ (on the ball $\|p_f\| = 1$), and then verify that p_f is indeed a certificate of non-$\mathcal{U}$-reachability for y_f. Since this stopping condition will never be satisfied if the target set $\mathcal{Y}_f$ is in fact $\mathcal{U}$-reachable, one might consider adding another condition based on how small an improvement is made from one step to another. As the functional J_d does not admit a minimiser (see Remark 3) in the non-reachable case, we use the stopping condition

$$\left\| \frac{p_{k+1}}{\|p_{k+1}\|} - \frac{p_k}{\|p_k\|} \right\| \leq \delta, \tag{16}$$

where δ is a small tolerance.

Carrying out a descent algorithm on J_d can be tackled by means of many optimisation techniques. For the following examples, we take advantage of the dual nature (see Appendix A) of the problem with the choice of functions (23), assuming $\mathcal{U}$ is convex. This allows us to use the Chambolle-Pock primal-dual algorithm [7]. It has the drawback of requiring a closed-form expression of two proximal operators associated with the functionals F^* and G, as defined in Appendix A. In general, if $\sigma_\mathcal{U}$ and $\sigma_{\mathcal{Y}_f}$ have closed-form formulae, so do those proximal operators.

4.2 The streetcar

Control problem. The following example is completely standard in optimal control theory. It can be found for example in [21, Chapter 1] and is concerned with the optimal control of the acceleration of a streetcar on a straight axis.

We will use this example to both illustrate and to validate our approach, since the reachable set and minimal times are known explicitly, see Appendix B.

We consider a streetcar moving on a graduated rectilinear axis. The initial position-velocity pair of the streetcar is assumed to be $(0,0)^T$ and the objective is to steer the system to some $y_f \in \mathbb{R}^2$ in minimal time. The control system reads

$$\begin{cases} \dot{y}_1(t) = y_2(t), \\ \dot{y}_2(t) = u(t), \end{cases} \tag{17}$$

which corresponds to the matrices

$$A := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B := \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{18}$$

For a fixed $M > 0$, the chosen constraint is given by

$$\mathcal{U} := \{u \in \mathbb{R},\ |u| \leq M\}.$$

Resolution method. First, we compute the support function

$$\forall u \in \mathbb{R}, \quad \sigma_{\mathcal{U}}(u) = M|u|,$$

which is a particular case of (5).

Here, we use the functional $J_{\mathrm{d},1}$ and the estimate given by Corollary 10. Given how simple $\sigma_{\mathcal{U}}$ and the control system are, we acknowledge that one could actually compute the functional J itself and only have to deal with round-off errors. We do not pursue this approach since we aim at analysing how prominent the discretisation errors may be.

The Jordan-Chevalley decomposition of A is straightforward in this case, since the matrix A is itself nilpotent, of index $\ell = 2$. In this case, we hence have $\mu = 0$, $\kappa(P) = 1$, $\ell = 2$, $Q_2(x) = 1 + x$, leading to the estimate

$$|J(p_f) - J_{\mathrm{d},1}(p_f)| \leq \frac{1}{2}\Delta t\ MT\|B\|\|A^* p_f\| Q_2(\|A\|T).$$

Results. To highlight the dependence of J with respect to the target y_f, we will temporarily rename $J(p_f)$ to $J(p_f; y_f)$. We give examples of targets $y_f \in \mathbb{R}^2$ that are certified to not be $\mathcal{U}$-reachable below, in the form of a computer-assisted theorem.

Theorem 16. *The following targets are not $\mathcal{U}$-reachable from $(0,0)$ in time $T = 1$, with $M = 1$*

$$y_1 = (0.1, 0.6)^T, \qquad y_2 = (0.5, 1.1)^T, \qquad y_3 = (0.3, 0)^T.$$

Indeed, the dual certificates

$$p_1 = (-0.77, 0.64)^T, \qquad p_2 = (0.29, 0.96)^T, \qquad p_3 = (0.85, -0.53)^T.$$

provide the intervals

$$J(p_1; y_1) \in [-0.0305, -0.0291] \quad J(p_2; y_2) \in [-0.0964, -0.0959], \quad J(p_3; y_3) \in [-0.0282, -0.0268].$$

The targets and dual certificates are plotted in Figure 2, along with the theoretically known reachable set.

Using the formula provided in Appendix B, the minimal times to reach y_1, y_2 and y_3 are computed to be slightly above 1.1656, 1.7480 and 1.0954, which means they are indeed not reachable.

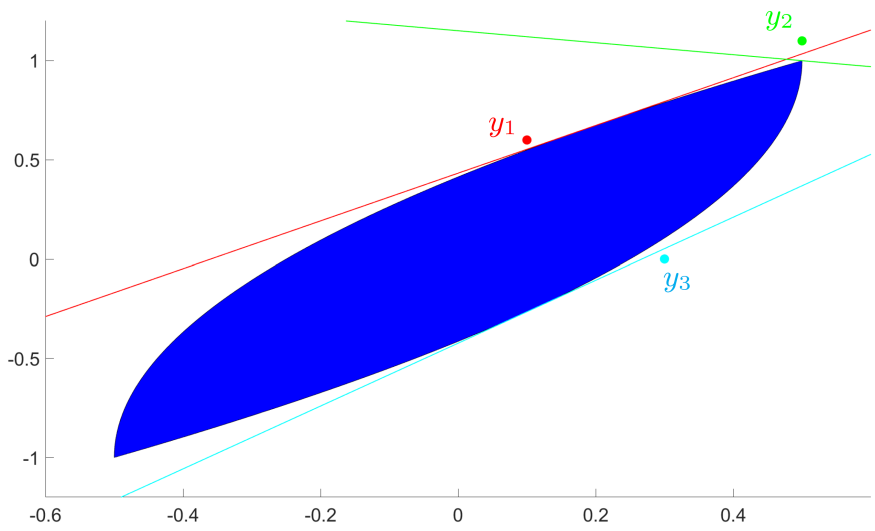

Figure 2: Non-$\mathcal{U}$-reachability of the targets from Theorem 16, together with the support hyperplane associated to their respective dual certificates, and the streetcar theoretical reachable set deduced from Appendix B.

4.3 Space rendezvous

Control problem. We here consider the 2-dimensional linearised Hill-Clohessy-Wiltshire equations, as defined in [8]. These equations model the motion of a follower spacecraft in the neighbourhood of a reference spacecraft $\big($at position $y_0 = (0,0,0,0)^T\big)$. The matrices underlying the control problem are

$$A := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{pmatrix}, \qquad B := \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{19}$$

Note that y_1, y_2 are positions and $y_3 = \dot{y}_1$, $y_4 = \dot{y}_3$ are the corresponding speeds.

We consider the following constraint set for fixed $M_2 > 0$, $M_\infty > 0$:

$$\mathcal{U} := \{u \in \mathbb{R}^2,\ \|u\|_2 \leq M_2,\ \|u\|_\infty \leq M_\infty\}, \tag{20}$$

hence we may take $M := \min(M_2, \sqrt{2}M_\infty)$.

Let us compute the support function $\sigma_{\mathcal{U}}$ in the case where $M_\infty \leq M_2 \leq \sqrt{2}M_\infty$, that we will consider hereafter. As illustrated in Figure 3, the constraint set is the intersection of a disk and a square. Observe that the boundary of $\mathcal{U}$ is the union of flat and circular parts, whose coordinates (x, y) of intersection points form the set

$$\mathcal{P} = \left\{\Big(\pm M_\infty, \pm\sqrt{M_2^2 - M_\infty^2}\Big)\right\} \bigcup \left\{\Big(\pm\sqrt{M_2^2 - M_\infty^2}, \pm M_\infty\Big)\right\}.$$

Let us write $\partial\mathcal{U} = \mathcal{F} \cup \mathcal{C}$, where $\mathcal{F}$ (resp. $\mathcal{C}$) denotes the union of all flat (resp. circular) parts of the boundary.

Let us fix $x \in \mathbb{R}^2$. We distinguish between two cases:

- if $(O; x) \cap \partial\mathcal{U} \subset \mathcal{C}$, meaning that $\frac{\|x\|_\infty}{M_\infty} \leq \frac{\|x\|_2}{M_2}$, then $M_2 \frac{x}{\|x\|_2} \in \mathcal{U}$ and using the Cauchy-Schwarz inequality, we get

$$\sigma_{\mathcal{U}}(x) \leq \sup_{y \in \mathcal{U}} \|x\|_2 \|y\|_2 = \left\langle x, M_2 \frac{x}{\|x\|_2} \right\rangle = M_2 \|x\|_2.$$

  We thus infer that $\sigma_{\mathcal{U}}(x) = M_2\|x\|_2$.

- Otherwise, $\sigma_{\mathcal{U}}(x)$ reads as the maximum of a linear (convex) function on a union of flat parts. We easily infer that $\sigma_{\mathcal{U}}(x) = \langle p_x, x\rangle$, where p_x denotes any point of the set $\operatorname{argmin}_{p\in\mathcal{P}} \|p - x\|_2$.

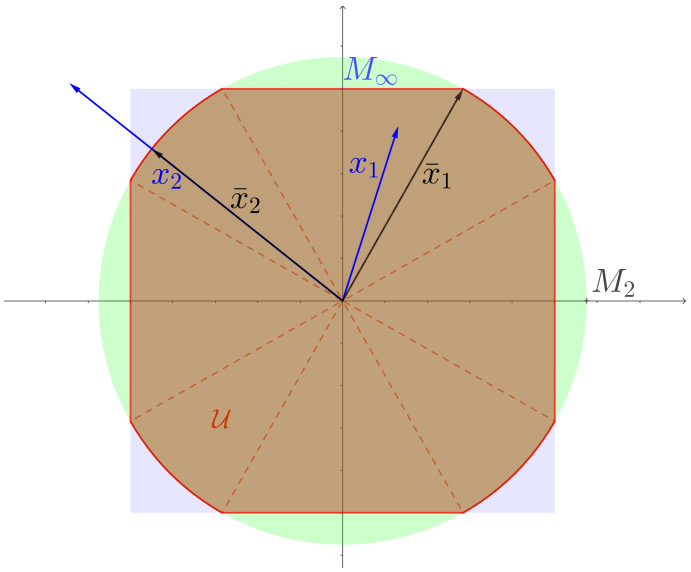

Figure 3: Construction of the support function for the rendezvous problem. One has in particular $\sigma_{\mathcal{U}}(x_i) = \langle x_i, \bar{x}_i\rangle$, $i = 1, 2$.

Resolution method. The Jordan-Chevalley decomposition of A is given by $A = D + N$ with

$$D := P \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix} P^{-1}, \quad N := P \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} P^{-1}, \quad P := \begin{pmatrix} 0 & -\frac{2}{3} & -1 & -1 \\ 1 & 0 & 2i & -2i \\ 0 & 0 & i & -i \\ 0 & 1 & 2 & 2 \end{pmatrix}.$$

Here, we also use the functional $J_{\mathrm{d},1}$ and the estimate given by Corollary 10. Using the corresponding notations, we have $\mu = 0$, and the index of the nilpotent matrix N is $\ell = 2$. Thus the corresponding estimate reads

$$|J(p_f) - J_{\mathrm{d},1}(p_f)| \leq \frac{1}{2}\Delta t\ MT\|B\|\|A^* p_f\|\kappa(P)Q_2(\|N\|T),$$

with $Q_2(x) = 1 + x$.

Results. Given a target $y_f \in \mathbb{R}^4$, we can derive a lower-bound on the minimal time needed to steer the system from $y_0 = (0, 0, 0, 0)^T$ to y_f. Proposition 7 ensures that we may indeed estimate the corresponding minimal time from below, using our approach. To compute this lower bound, we apply a bisection algorithm over the set of positive real numbers, starting from a predefined interval $[t_{\mathrm{inf}}, t_{\mathrm{sup}}]$, and expanding it by multiplying its length by 2 until we cannot prove the non-reachability in time t_{sup}, and we can prove it in time t_{inf}. Then, the standard bisection method applies until the interval is reduced to the desired length.

First, we consider the time-minimal control problem of steering the system from $y_0 = (0, 0, 0, 0)^T$ to some other position at 0 speed, i.e., $y_f = (y_1, y_2, 0, 0)^T$ for various values of $(y_1, y_2) \in \mathbb{R}^2$. Since the control problem is linear and the constraints centrally symmetric (i.e., $\mathcal{U} = -\mathcal{U}$), if y_f is reachable in time $T > 0$, so is $-y_f$. This translates into the identity $J(p_f; y_f) = J(-p_f; -y_f)$, allowing us to focus our computations on the right half-plane.

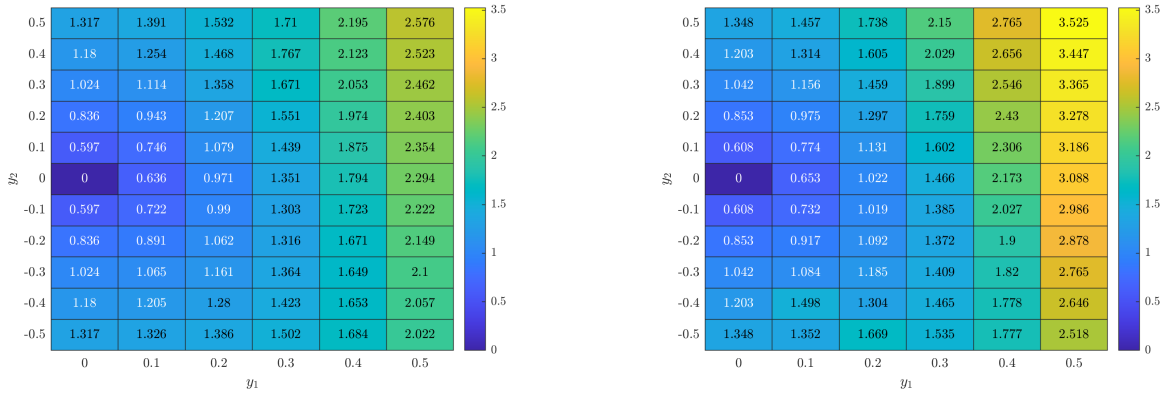

Figure 4: Estimates of the minimal time for reachability of various targets at speed 0 for the spacecraft rendezvous control problem. Certified lower bounds (left panel (a)) versus minimal times outputted by Gekko Optimization Suite [4] (right panel (b)).

Using the bounds $M_2 = 1.15$ and $M_\infty = 1$, we obtain the certified lower bounds on the minimal-time shown on Figure 4(a). For conciseness, we do not provide the corresponding dual certificates. For comparison purposes, the minimal times computed using the Python package Gekko [4] are presented in Figure 4(b). Note that Gekko does not control discretisation bounds nor roundoff errors, hence the corresponding estimates are by no means certified.

Computation times. As is common, our certified method comes at the price of increased computation times: each step of the bisection algorithm is rather fast (about 30 seconds on a standard desktop computer), but depending on parameters and how good the initial guess is, the number of iterations of the bisection algorithm may go from 3-4 to 10-15 iterations, whereas Gekko's method computes one approximation of the minimal time in about 10 seconds.

Assuming that Gekko produces reliable estimates, the accuracy of our method seems to decrease the further the target y_f is from y_0, going from about 1.8% to 37%. This can be explained as follows: our computations were made with a fixed number of time steps, namely $N_t = 20,000$; hence the higher the theoretical minimal time is, the harder it is to establish a tight lower-bound. Increasing N_t allows for a more precise approximation: for example, for $y_f = (0.5, 0.5, 0, 0)^T$, with $N_t = 400,000$, the dual certificate $p_f = (0.874, 0.0914, -0.3008, 0.3704)^T$ proves the bound $t_{\min} \geq 3.4$, which is about 3.7% away from Gekko's approximation.

On the other hand, Gekko seems to produce what might be artefacts (points $(0.1, -0.4)^T$ and $(0.2, -0.5)^T$), while our computed certified lower bounds remain smooth.

More complex unsafe set $\mathcal{Y}_f$. Now we look at the case where ones wants to avoid a given spherical object *in space*, motionless in the considered referential, regardless of the speed. In other words, for a fixed choice of $(z_1, z_2) \in \mathbb{R}^2$, and $\varepsilon > 0$, we consider

$$\mathcal{Y}_f = \left\{(y_1, y_2, y_3, y_4) \in \mathbb{R}^4,\ \|(y_1 - z_1, y_2 - z_2)\|_{\mathbb{R}^2} \leq \varepsilon\right\}, \tag{21}$$

In that case $\mathcal{Y}_f$ is unbounded; letting $z := (z_1, z_2, 0, 0)$, the support function of $\mathcal{Y}_f$ can be computed to be

$$\sigma_{\mathcal{Y}_f} : x \longmapsto \langle z, x\rangle + \varepsilon\|x\|_2 + \delta_{\{y\in\mathbb{R}^4,\ y_3=y_4=0\}}(x),$$

where we use the convex analytic notation $\delta_C(x) = 0$ if $x \in C$ and $+\infty$ instead. We prove below a certified result for one such example.

Theorem 17. *Take $z_1 = z_2 = 0.5$, $\varepsilon = 0.1$, $M_2 = 1.15$, $M_\infty = 1$ and $T = 1$. Then $\mathcal{Y}_f$ is not*

$\mathcal{U}$-reachable from $(0,0,0,0)^T$ in time T. Indeed, we find

$$J(p_f) \ \in \ [-0.1146, -0.0717], \quad \textit{with} \quad p_f = (0.62, 0.78, 0, 0)^T.$$

Moreover, since $y_0 = (0,0,0,0)^T$, for any $t \in [0,T]$, $\mathcal{Y}_f$ is not $\mathcal{U}$-reachable from $(0,0,0,0)^T$ in time t.

4.4 Negative semi-definite Jacobi matrices

Control problem. In this section, we report on results for some randomly generated Jacobi matrices, with varying state dimensions n. That is, we consider matrices of the form

$$A = \begin{pmatrix} a_1 & c_1 & 0 & \dots & 0 \\ c_1 & a_2 & c_2 & \ddots & \vdots \\ 0 & c_2 & a_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & c_{n-1} \\ 0 & \dots & 0 & c_{n-1} & a_n \end{pmatrix}. \tag{22}$$

with $a = (a_1, \dots, a_n) \in \mathbb{R}^n$, $c = (c_1, \dots, c_{n-1}) \in \mathbb{R}^{n-1}$.

These matrices are real symmetric, and up to our knowledge, no closed-form expressions are known for their eigenvalues and eigenfunctions, except in the specific case where the c_i's are all equal. Hence, for large values of n, diagonalising A becomes intractable. Even if it were accessible, it would be prone to numerical errors and we are not aware of any software that does produce such a diagonalisation within interval arithmetic.

We generate such a matrix in the following way: let $K > 0$ and $L > 0$. We draw the c_i's uniformly in $[-K, K]$. Then, we draw the a_i's uniformly in $(-2K - L, -2K]$. Thanks to the Gershgorin circle theorem, the resulting matrix is negative semi-definite.

We consider a single control u, thus $m = 1$. The corresponding matrix $B \in \mathbb{R}^{n\times 1}$ is $B = (1, \dots, 1)^T$. For a fixed $M > 0$, the constraint set is given by

$$\mathcal{U} = \{u \in \mathbb{R}, \ |u| \leq M\},$$

for which we have $\sigma_{\mathcal{U}}(u) = M|u|$. The target y_f is chosen randomly, with i.i.d entries uniformly in $[-1, 1]$, then normalised such that $\|y_f\| = 0.05$.

Resolution method. Under the assumptions mentioned above, all eigenvalues of A are nonpositive according to the Gershgorin circle theorem.

As a result, we are dealing with negative semi-definite matrices, enabling us to use estimates coming from Proposition 14 upon using the Euler implicit scheme to approximate the matrix exponential.

Results. In the following example, we shall take $M = 1$, $T = 1$, $N_t = 1{,}000$ and $y_0 = 0$.

Random targets. For each chosen dimension n, we generate 200 experiments with a target y_f of fixed norm $\|y_f\| = 0.05$, and a random matrix A drawn as explained previously (with $K = 2$, $L = 0.1$). More precisely, we run our descent algorithm to try and prove the non-$\mathcal{U}$-reachability of y_f from y_0 in time T. The following table shows the resulting means and standard deviations for the midpoint and size of the obtained intervals around $J(\frac{p_f}{\|p_f\|})$ where p_f is the last iterate of the optimisation algorithm.

Recalling the notation $E_{\mathcal{U}} = \{u \in E, \ t \in (0,T), \ u(t) \in \mathcal{U} \text{ for a.e. } t \in (0,T)\}$, it will be convenient to report values for the two terms involved in the definition of J, namely

$$J\Big(\frac{p_f}{\|p_f\|}\Big) = \sigma_{E_{\mathcal{U}}}\Big(L_T^* \frac{p_f}{\|p_f\|}\Big) - \Big\langle y_f, \frac{p_f}{\|p_f\|}\Big\rangle.$$

Indeed, once p_f and y_f are fixed, only the second term depends on the target y_f. Assume that $\langle y_f, \frac{p_f}{\|p_f\|}\rangle > 0$ and $J(\frac{p_f}{\|p_f\|}) > 0$ (so nothing is known about the $\mathcal{U}$-reachability of y_f). Then one can dilate y_f, i.e. change y_f to λy_f with $\lambda > 0$, and obtain a value $\lambda^\star > 0$ such that λy_f is not $\mathcal{U}$-reachable from 0 in time T, for any $\lambda > \lambda^\star$.

We also display below the proportion of targets which are proved to be non-reachable.

n	$\langle y_f, \frac{p_f}{\|p_f\|}\rangle$	Midpoints of $\sigma_{E_\mathcal{U}}\big(L_T^* \frac{p_f}{\|p_f\|}\big)$			Radii of $\sigma_{E_\mathcal{U}}\big(L_T^* \frac{p_f}{\|p_f\|}\big)$			Proportion of guaranteed non-reachable targets
		Mean	Median	Standard deviation	Mean	Median	Standard deviation	
2	0.0316	0.0027	0.0019	0.0063	0.0083	0.0082	0.0023	0.845
5	0.0369	0.0057	0.0036	0.0053	0.0148	0.0147	0.0026	0.91
10	0.0426	0.0043	0.0032	0.0033	0.0210	0.0209	0.0030	0.965
20	0.0457	0.0040	0.0035	0.0021	0.0288	0.0291	0.0032	0.97
50	0.0483	0.0067	0.0065	0.0026	0.0462	0.0461	0.0034	0.145

As the dimension n grows, one should expect that the proportion of final states y_f of norm $\|y_f\| = 0.05$ that are non-$\mathcal{U}$-reachable (from 0 in time T) should approach 1, as we keep a single control ($m = 1$). In fact, this is what is seen up until $n = 20$. Then, when is the dimension is increased to $n = 50$, this proportion drops to about 15% if the tolerance δ defining the stopping criterion (16) is kept to its initial value $\delta = 10^{-5}$. By diminishing δ to $\delta = 5.10^{-7}$, we partially mitigate this problem, guaranteeing the non-$\mathcal{U}$-reachability of about 52% of states y_f. Obtaining even higher values becomes computationally prohibitive, making the case of dimension $n = 100$ intractable.

If, however, we increase the norms of targets y_f from 0.05 to 0.1, then a tolerance of $\delta = 10^{-5}$ is enough to certify that almost all such targets are non-$\mathcal{U}$-reachable, even in the case $n = 100$.

The main takeaway is that the tolerance δ should be adapted to the problem dimensions, and to how close the target of interest is to the unknown reachable set. Another degree of liberty is to increase the number of time steps N_t, which leads to a reduction of the error term at the expense of increased computation time. Regardless, these results suggest that our approach suffers from a sort of curse of dimensionality.

Size of the reachable set. We then try to estimate the size of the reachable set from $y_0 = 0$, i.e., $L_T E_\mathcal{U}$. For a given dimension n, we randomly choose a fixed matrix A in the same way as before. Then, we draw 1000 vectors $\tilde{p}_f$ at random on the unit sphere of $\mathbb{R}^n$, and report the statistics obtained for (the intervals) $\sigma_{L_T E_\mathcal{U}}(\tilde{p}_f) = \sigma_{E_\mathcal{U}}(L_T^* \tilde{p}_f)$.

n	Midpoints of $\sigma_{E_\mathcal{U}}(L_T^* \tilde{p}_f)$			Radii of $\sigma_{E_\mathcal{U}}(L_T^* \tilde{p}_f)$		
	Mean	Median	Standard deviation	Mean	Median	Standard deviation
2	0.3527	0.3874	0.1837	0.0087	0.0089	0.0014
5	0.3487	0.3112	0.2138	0.0138	0.0138	0.0022
10	0.3305	0.2783	0.2163	0.0204	0.0201	0.0027
20	0.2982	0.2523	0.2056	0.0292	0.0292	0.0032
50	0.3305	0.2839	0.2131	0.0467	0.0467	0.0037
100	0.3427	0.2983	0.2217	0.0651	0.0651	0.0032

As can be seen, although the midpoints of intervals are rather constant, the error term steadily increases, which leads to more difficult proofs of non-reachability. As already mentioned, one could increase N_t to reduce errors.

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A Convex analytic interpretation

Here, we make the additional assumption that the constraint set $\mathcal{U}$ is not only compact, but also convex. For all the following definitions and results, we refer e.g. to [3].

We let H be a Hilbert space. We recall that a function $f : H \to \mathbb{R} \cup \{+\infty\}$ is said to be *proper* if it is not identically equal to $+\infty$.

Definition-Proposition 18. *We define*

$$\Gamma_0(H) := \{f : H \to \mathbb{R} \cup \{+\infty\},\ f \text{ convex, lower semi-continuous and proper}\}.$$

We denote by $f^* : H \to \mathbb{R} \cup \{+\infty\}$ *the convex conjugate of* $f \in \Gamma_0(H)$

$$f^* : y \mapsto \sup_{x \in H} \langle x, y\rangle - f(x).$$

Furthermore, f^* *belongs to* $\Gamma_0(H)$.

Definition 19. *For* $C \subset H$ *a nonempty closed convex subset, we denote by* $\delta_C : H \to \mathbb{R} \cup \{+\infty\}$ *the convex indicator function of* C

$$\delta_C : x \mapsto \begin{cases} 0 & \text{if } x \in C \\ +\infty & \text{if } x \notin C. \end{cases}$$

We have $\delta_C \in \Gamma_0(H)$.

By definition, note that $\delta_C^* = \sigma_C$.

Theorem 20 (Weak and strong duality)**.** *Let* E, X *be two Hilbert spaces,* $K \in L(E, X)$, $F \in \Gamma_0(E)$, *and* $G \in \Gamma_0(X)$. *Then we have the following so-called weak duality*

$$\inf_{x \in E} F(x) + G(Kx) \geq - \inf_{y \in X} F^*(K^*y) + G^*(-y).$$

If in addition there exists $p_f \in X$ *such that* F^* *is continuous at* $L_T^* p_f$, *then strong duality holds, i.e.,*

$$\inf_{x \in E} F(x) + G(Kx) = - \inf_{y \in X} F^*(K^*y) + G^*(-y).$$

Fenchel-Rockafellar interpretation of our approach. Since the compact constraint set $\mathcal{U}$ is assumed to be convex, so is the set

$$E_{\mathcal{U}} = \{u \in E,\ t \in (0,T),\ u(t) \in \mathcal{U} \text{ for a.e. } t \in (0,T)\}.$$

An alternative approach to the one leading to Proposition 2 is then to remark that $\mathcal{Y}_f$ is $\mathcal{U}$-reachable from y_0 in time T if and only if

$$\exists u \in E, \quad \delta_{E_{\mathcal{U}}}(u) + \delta_{\mathcal{Y}_f - e^{TA} y_0}(L_T u) = 0,$$

in other words if and only if

$$\inf_{u \in E} \delta_{E_{\mathcal{U}}}(u) + \delta_{\mathcal{Y}_f - e^{TA} y_0}(L_T u) = 0.$$

Note that the above functional takes at most two values, 0 and $+\infty$. Denoting

$$F := \delta_{E_{\mathcal{U}}}, \qquad G := \delta_{\mathcal{Y}_f - e^{TA} y_0}, \tag{23}$$

we have $F \in \Gamma_0(E)$, $G \in \Gamma_0(\mathbb{R}^n)$ and we find that

$$F^*(L_T^* p_f) + G^*(-p_f) = \sigma_{E_{\mathcal{U}}}(L_T^* p_f) - \sigma_{\mathcal{Y}_f - e^{TA} y_0}(-p_f) = J(p_f).$$

Furthermore, it is easily seen that F^* is continuous at $0 = L_T^* 0$. Thus we can apply Theorem 20 to obtain the strong duality

$$\inf_{u\in E} \delta_{E_{\mathcal{U}}}(u) + \delta_{\mathcal{Y}_f - e^{TA}y_0}(L_T u) = -\inf_{p_f\in\mathbb{R}^n} J(p_f).$$

In particular, we see that if there exists p_f such that $J(p_f) < 0$, then $\inf_{p_f\in\mathbb{R}^n} J(p_f) < 0$ (in which case this infimum even equals $-\infty$), hence the infimum on the left-hand side equals $+\infty$, and $\mathcal{Y}_f$ is not $\mathcal{U}$-reachable from y_0 in time T. Conversely, if $\mathcal{Y}_f$ is not $\mathcal{U}$-reachable from y_0 in time T, the left-hand side equals $+\infty$, which leads to $\inf_{p_f\in\mathbb{R}^n} J(p_f) = -\infty$, so that there exists $p_f \in \mathbb{R}^n$ satisfying $J(p_f) < 0$.

B Minimal time for the streetcar example

Proposition 21. *Let $(x_f, y_f) \in \mathbb{R}^2$. The minimal time to steer System (17) from $(0,0)$ to (x_f, y_f) reads*

$$T = \frac{-sy_f + 2\sqrt{\frac{1}{2}y_f^2 + sMx_f}}{M}, \quad \text{with } s = \operatorname{sign} f(x_f, y_f),$$

using the convention $\operatorname{sign}(0) = 0$*, where $f : \mathbb{R}^2 \to \mathbb{R}$ is given by*

$$f(x,y) = x - \frac{1}{2M}y^2 \operatorname{sign}(y).$$

Proof. Let T be the optimal time steering System (17) from $(0,0)$ to (x_f, y_f). According to [21, Chapter 1], it is well-known that optimal controls are bang-bang equal a.e. to M or $-M$, with at most one switch, on the so-called switching locus defined by the implicit equation $f(x,y) = 0$.

More precisely, if $s < 0$, then the optimal control is $u = M\mathbb{1}_{(0,t_0)} - M\mathbb{1}_{(t_0,T)}$, where $t_0 \geq 0$ is the switching time, in other words the first time such that $f(x(t), y(t)) = 0$. Conversely, if $s > 0$, then $u = -M\mathbb{1}_{(0,t_0)} + M\mathbb{1}_{(t_0,T)}$. Easy but lengthy computations yield

- If $f(x_f, y_f) = 0$, then for every $t \in [0,T]$, one has

$$y(t) = y_0 - Mt\operatorname{sign}(y_f) \quad \text{and} \quad x(t) = x_f - y_f t - \frac{1}{2}Mt^2\operatorname{sign}(y_f).$$

- Conversely, if $f(x_f, y_f) \neq 0$, then for every $t \in [0,T]$, one has

$$\begin{aligned} y(t) &= (-y_f - sMt)\mathbb{1}_{(0,t_0)} + (y_f + sM(t-2t_0))\mathbb{1}_{(t_0,T)} \\ x(t) &= \big(x_f - y_f t - \tfrac{1}{2}sMt^2\big)\mathbb{1}_{(0,t_0)} + \big(x_f - y_f t + sM(\tfrac{1}{2}t^2 - 2t_0 t + t_0^2)\big)\mathbb{1}_{(t_0,T)}. \end{aligned}$$

To conclude, it is important to notice that if $s \neq 0$, then $\operatorname{sign}(y(t_0)) = s$, which can be easily seen by distinguishing between several cases, depending on the sign of y_f and s.

To conclude, it remains to compute the switching time t_0. We claim that if $f(x_0, y_0) \neq 0$, then

$$t_0 = \frac{1}{M}\left(-s\,y_f + \sqrt{\frac{1}{2}y_f^2 + s\,Mx_f}\right).$$

Indeed, t_0 is characterised by the equation $f(x(t_0), y(t_0)) = 0$, which rewrites as the second order polynomial equation in the variable t_0:

$$0 = \left(x_f - s\,\frac{1}{2M}y_f^2\right) - y_f(1+s^2)t_0 - s\,Mt_0^2.$$

Furthermore, the discriminant of this polynomial is positive. It follows that $y(T) = -y_f + sM(T - 2t_0)$ and therefore, $T = \frac{s}{M}y_f + 2t_0$. The expected conclusion follows. □

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