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Online parameter estimation for a bilinear control problem

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Motivation (Bloch-Torrey equation for MRI)

State: $z^\star$ (magnetisation), control $q(t) = (u(t), g(t))$ with radio-frequency pulses $u(t)$ and gradients $g(t)$, parameters $T_1^\star$, $T_2^\star$, $z_{\text{eq}}^\star$.

$$\partial_t z^\star + \gamma z^\star \times \underbrace{\begin{pmatrix} u_x(t) \\ u_y(t) \\ B_0 + g(t)\cdot \end{pmatrix}}_{\text{excitation}} - \underbrace{D\,\Delta z^\star}_{\text{diffusion}} + \underbrace{\text{diag}\left(\tfrac{1}{T_2^\star}, \tfrac{1}{T_2^\star}, \tfrac{1}{T_1^\star}\right)}_{\text{relaxation}} z^\star = \begin{pmatrix} 0 \\ 0 \\ z_{\text{eq}}^\star / T_1^\star \end{pmatrix}.$$

Excitation → **Acquisition** → **Reconstruction**

Excitation: choose and apply the control $q = (u, g)$

Acquisition: measure $y = Rz^\star$

Reconstruction: recover estimators z, T_1, T_2, z_{eq}

Abstract framework & objectives

Main equation:

$$\partial_t z^\star + [H_0(z^\star) - H_1]\theta^\star + [C_0(q) - C_1] z^\star = 0.$$

Partial state observation:

$$y = Rz^\star.$$

Objectives:

- reconstruct both state $z^\star(t)$ and parameter $\theta^\star$,
- do so quickly and precisely,
- design a control q optimising the reconstruction.

Online parameter estimation

Estimators z of $z^\star$ and $\theta(t)$ of $\theta^\star$:

$$\partial_t z + [H_0(z) - H_1]\theta + [C_0(q) - C_1] z = \Psi^z(z, \theta, y),$$
$$\partial_t \theta = \Psi^\theta(z, \theta, y).$$

Advantages:

- reconstruction starts during acquisition (quicker feedback),
- enables observation-based controls $q(t, x, y)$.

State of the art:

- Adaptive Lyapunov – Persistence of Excitation methods: [1, 3, 4].
- Partial-observation online estimators: [3, 4].
- Nonlinear inverse and input-design approaches: [2, 5, 6].

Estimator construction

Estimators $z^\perp$ (of $z^\star - R^\dagger y$) and θ (of $\theta^\star$):

$$\begin{cases} \bar{z} = R^\dagger y + z^\perp, \\ \partial_t z^\perp + R^\perp\big(H_0(\bar{z})\theta + C_0(q)\bar{z}\big) = \Psi^\perp, \\ \partial_t \theta = \Psi^\theta. \end{cases}$$

Differentiate the observation:

$$\partial_t y = R\partial_t z^\star = -R\big([H_0(z^\star) - H_1]\theta^\star + [C_0(q) - C_1] z^\star\big),$$

obtaining the new observable residual $r = R\,\partial_t \bar{z} - \partial_t y$ satisfying

$$r = R\big([H_0(\bar{z}) - H_1](\theta - \theta^\star) + [\tilde{H}_0(\theta^\star) + C_0(q) - C_1](\bar{z} - z^\star)\big).$$

Hypotheses

- *Regularity hypotheses* on H_0, H_1, C_0, C_1: guarantee well-posedness of equations and estimators.
- *Initial error smallness*: necessary condition for estimator convergence for nonlinear systems
- *Persistence of excitation*, to guarantee that the parameter is identifiable: for a given control q, there exist $\tau > 0$ and $\varepsilon > 0$ such that for all $T > 0$,

$$\int_T^{T+\tau} \|r\|^2 \,\mathrm{d}t \;\geq\; \varepsilon_q \,\|\theta - \theta^\star\|^2.$$

Results

Proposition

For a gain $\mu(t) > 0$ large enough, choose

$$\begin{cases} \Psi^\perp = -\mu\,[\tilde{H}_0(\theta) + C_0(q) - C_1]^* r, \\ \Psi^\theta = -\mu\,[H_0(\bar{z}) - H_1]^* r. \end{cases}$$

Under aforementioned hypotheses, we obtain $0 < \alpha < 1$, such that for all $t > 0$

$$\|\theta(t + T) - \theta^\star\|^2 \leq \alpha\,\|\theta(t) - \theta^\star\|^2.$$

Toy numerical example

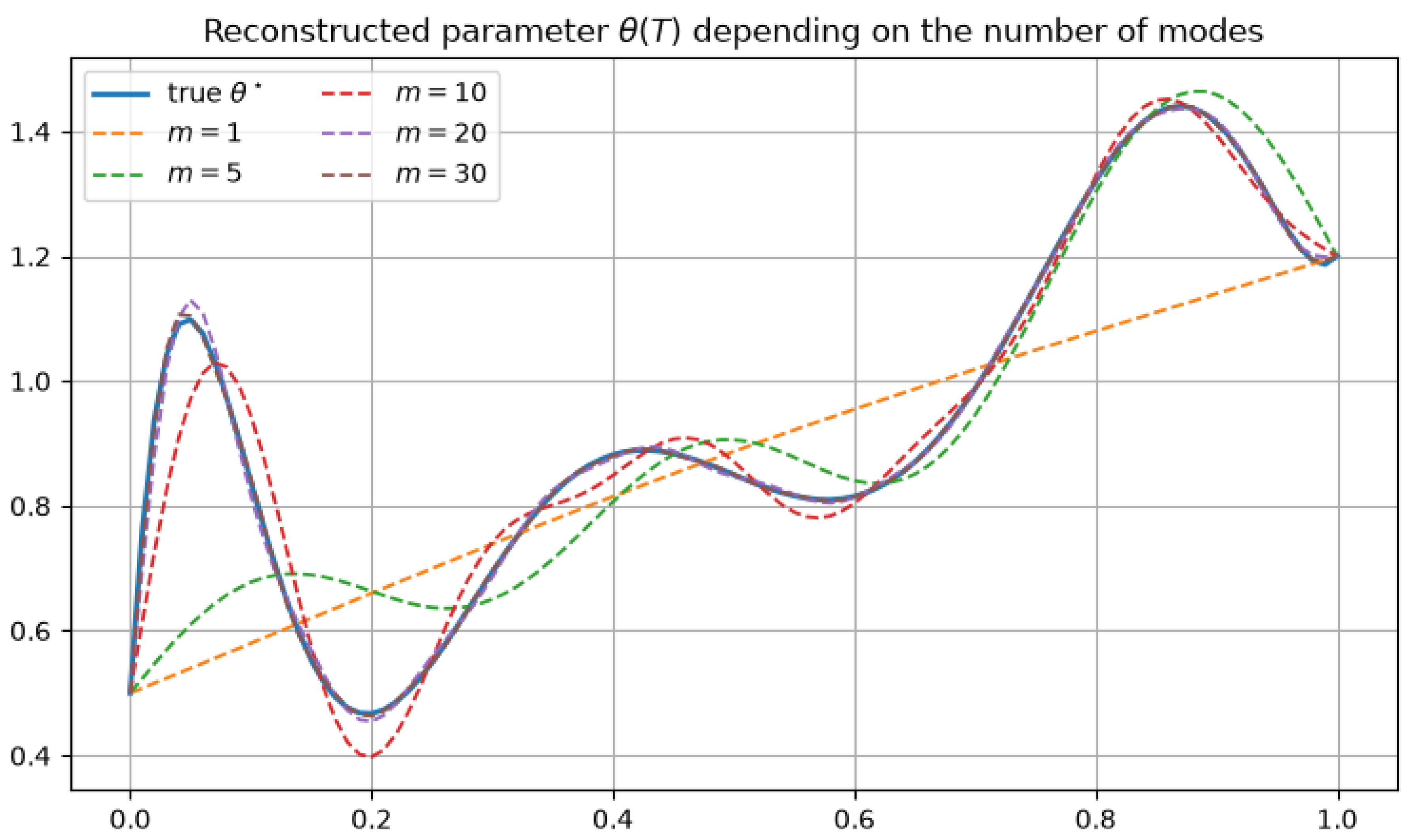
Uncontrolled $1D$ heat-equation:

$$\begin{cases} \partial_t z^\star - \Delta z^\star + \theta^\star z^\star = b z^\star & \text{on } (0,1), \\ z^\star = 0 & \text{on } \{0,1\}, \end{cases}$$

with spectral partial observation:

$$y = 2\sum_{k=1}^{m} \langle \sin(k\pi\cdot),\, z^\star \rangle \sin(k\pi\cdot)$$

and knowledge of $\theta^\star(0)$ and $\theta^\star(1)$. For $T = 1$ and $\mu(t) = 10$, convergence is partial and depends on the number of modes:



Outlook

How should one choose the control to optimise the reconstruction?

- maximise the associated observability constant,
- consider open-loop vs. observation-dependent controls.

Jointly, can one lessen the persistence of excitation condition to make it estimator-independent?

References

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