Online parameter estimation for a bilinear control problem

Ivan HASENOHR

Joint work with Barbara Kaltenbacher

French-Romanian Colloquium on Applied Mathematics

Nancy, June 22nd, 2026

10.55776/F100800

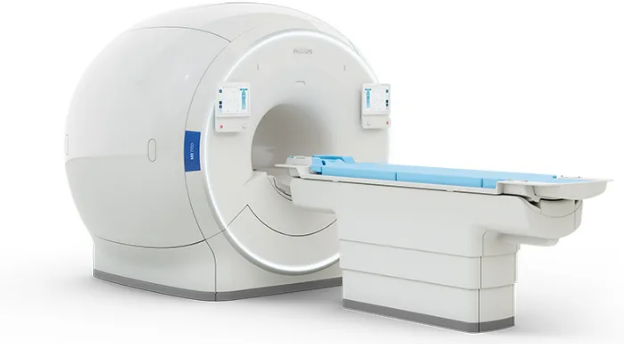
Summary

Summary

1 Motivation

2 Estimator construction & results

3 Conclusion & outlook



Bloch-Torrey

Bloch-Torrey equation:

$$\partial_t z^\star + q \times z^\star + \alpha z^\star + \Delta z^\star = z^\star_{\mathrm{eq}}.$$

- $z^\star$: magnetisation state.
- q: excitation / control.
- $\alpha, z^\star_{\mathrm{eq}}$: unknown relaxation-type time-independent parameter.

From excitation to reconstruction

Bloch-Torrey equation:

$$\partial_t z^\star + q \times z^\star + \alpha z^\star + \Delta z^\star = z^\star_{\mathrm{eq}}.$$

Excitation → **Acquisition** → **Reconstruction**

choose and apply the control q

measure the observation $y = Rz^\star$

recover estimators z, α, z_{eq}

Framework & objectives

System:

$$\begin{cases} \partial_t z^\star + H_0(z^\star)\theta^\star + C(q)z^\star = 0, \\ y = Rz^\star. \end{cases} \tag{$\mathcal{S}$}$$

Objectives:

- reconstruct the parameter $\theta^\star$,
- do so quickly and precisely,
- design a control q to optimise the reconstruction.

Summary

1. Motivation
2. Estimator construction & results
3. Conclusion & outlook

Online parameter estimation

$$
\begin{aligned}
\text{System:}\quad & \partial_t z^\star + H_0(z^\star)\theta^\star + C(q)z^\star = 0,\\
\text{Observation:}\quad & y = Rz^\star,\\
\text{State estimator:}\quad & \partial_t z + H_0(z)\theta + C(q)z = \Psi^z(z,\theta,y),\\
\text{Parameter estimator:}\quad & \partial_t \theta = \Psi^\theta(z,\theta,y).
\end{aligned}
$$

Advantages:

- starts reconstruction during acquisition

$$
\Longrightarrow \begin{cases} \text{quicker reconstruction} \\ \text{good initial condition for another reconstruction algorithm} \end{cases}
$$

- allows for observation-based excitation $q(t,x,y)$.

State of the art

Classical (control-less) approach[1]: estimators θ and z such that

$$\begin{cases} \partial_t z + H_0(z)\theta = H_0(Re^z)\theta - \lambda R^* R e^z e^z, \\ \partial_t \theta = H_0(z)^* R e^z, \\ e^z = z - z^\star, \\ e^\theta = \theta - \theta^\star. \end{cases}$$

Energy estimates:

$$\Gamma = \frac{1}{2}\left(\|e^z\|^2 + \|e^\theta\|^2\right).$$

$$\partial_t \Gamma = -\lambda(t)\|Re^z\|^2 + \left\langle H_0(Re^z)e^\theta - H_0(R^\perp e^z)\theta^\star, e^z\right\rangle - \left\langle R^\perp e^z, H_0(z)e^\theta\right\rangle.$$

[1] Baumeister et al. 1997; Kügler 2010; Boiger and Kaltenbacher 2016.

Estimator construction

$$\begin{cases} \partial_t z^\star + H_0(z^\star)\theta^\star + C(q)z^\star = 0, \\ y = Rz^\star. \end{cases} \tag{$\mathcal{S}$}$$

Differentiate the observation: $\partial_t y = R\partial_t z^\star = -R\big[H_0(z^\star)\theta^\star + C(q)z^\star\big]$.

New information: $r = \partial_t(Rz^\star - y) = R\Big(\big[\tilde{H}_0(\theta^\star) + C(q)\big]e^z + H_0(\bar{z})e^\theta\Big)$.

New estimators $z^\perp$ and θ of $R^\perp z^\star$ and $\theta^\star$:

$$\begin{cases} \bar{z} = y + z^\perp \\ \partial_t z^\perp + R^\perp\Big(H_0(\bar{z})\theta + C(q)\bar{z}\Big) = \Psi^\perp \\ \partial_t \theta = \Psi^\theta. \end{cases}$$

Literature: similar ideas in Lepsien et al. 2025.

Sketch of proof

For $\mu(t) > 0$ to be chosen, take

$$\begin{cases} \Psi^{\perp} = \mu\left[\tilde{H}_0(\theta) + C(q)\right]^* r, \\ \Psi^{\theta} = \mu\left[H_0(\bar{z})\right]^* r. \end{cases}$$

Then:

$$\begin{aligned} \partial_t \Gamma = & -\left\langle H_0(e^z)\theta^\star + H_0(\bar{z})e^\theta, e^z \right\rangle \\ & -\mu\langle\left[\tilde{H}_0(\theta^\star) + C(q)\right]e^z + H_0(\bar{z})e^\theta, RH_0(e^z)e^\theta\rangle \\ & -\mu\|r\|^2. \end{aligned}$$

Hypotheses:

- Continuity/coercivity of the operators involved.
- Smallness of initial parameter error: $\theta_0 \simeq \theta^\star$.
- Persistence of excitation: $\exists\tau > 0$, $\exists\varepsilon > 0$, $\forall T > 0$,

$$\int_T^{T+\tau} \|r\|^2 \geq \varepsilon\|e^\theta\|^2.$$

Summary

What about the control?

The persistence of excitation $\exists\tau > 0$, $\exists\varepsilon > 0$, $\forall T > 0$,

$$\int_T^{T+\tau} \left\| R\left(\left[\tilde{H}_0(\theta^\star) + C(q) \right] e^z + H_0(\bar{z}) e^\theta \right) \right\|^2 \geq \varepsilon \| e^\theta \|^2.$$

boils down to a control-dependant observability inequality:

$$\int_0^T \left\| R\left(\left[\tilde{H}_0(\theta^\star) + C(q) \right] e_q^z + H_0(\bar{z}_q) e_q^\theta \right) \right\|^2 \geq \varepsilon_q \| e_q^\theta \|^2.$$

Control optimisation must be done:

- to maximise the observability constant ε_q,
- with open-loop or observation-dependant controls.

Conclusion & outlook

We have:

- a nice general framework,
- an online method that should work.

On the way:

- convergence proof, including noise robustness,
- control design,
- numerical implementation.

Future works:

- what of differentiating the observation multiple times?
- Can we consider more general systems?

Thank you for listening!

ALPEN-ADRIA UNIVERSITÄT KLAGENFURT

FWF

10.55776/F100800

Baumeister, Johann, W Scondo, MA Demetriou, and IG Rosen (1997). "On-line parameter estimation for infinite-dimensional dynamical systems". In: *SIAM Journal on Control and Optimization* 35.2, pp. 678–713.

Boiger, Romana and Barbara Kaltenbacher (2016). "An online parameter identification method for time dependent partial differential equations". In: *Inverse Problems* 32.4, p. 045006.

Kügler, Philipp (2010). "Online parameter identification without Riccati-type equations in a class of time-dependent partial differential equations: an extended state approach with potential to partial observations". In: *Inverse Problems* 26.3, p. 035004.

Lepsien, Arthur, Philipp Kügler, and Alexander Schaum (2025). "On the Structural Analysis and Optimal Input Design for Joint State and Parameter Estimation". In: *IEEE Access* 13, pp. 211944–211959.