Online parameter reconstruction of a bilinear control problem

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Summary

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MRI & Bloch-Torrey

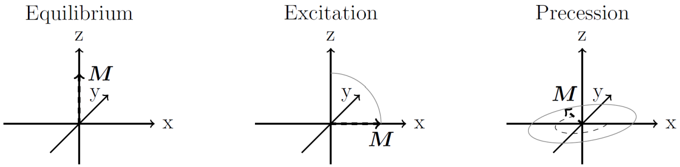
Bloch-Torrey equation:

$$\dot{z}^\star + q \times z^\star + \operatorname{diag}\left(\frac{1}{T_2}, \frac{1}{T_2}, \frac{1}{T_1}\right) z^\star + \Delta z^\star = z^\star_{\mathrm{eq}}.$$

Observations:

$$y(t) = \left(\langle c_j, z^\star \rangle\right)_{j \in \{1,\dots,n\}}.$$

MRI & Bloch-Torrey



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Framework & Objectives

System :

$$\begin{cases} \dot{z}^\star + H_0(z^\star)\theta^\star + C(q)z^\star = 0, \\ y = Rz^\star. \end{cases} \tag{$\mathcal{S}$}$$

Objectives:

- reconstruct the parameter $\theta^\star$
- do so quickly and precisely $\implies$ online method
- design a (feedback?) control q to optimise the reconstruction

Literature

Classical approach (Baumeister et al. 1997; Kügler 2010; Boiger and Kaltenbacher 2016): estimators θ and z such that

$$\begin{cases} \dot{z} + H_0(z)\theta + C(q)z = H_0(Re^z)\theta - \lambda R^* Re^z \\ \dot{\theta} = H_0(z)^* Re^z, \\ e^z = z - z^\star \\ e^\theta = \theta - \theta^\star. \end{cases}$$

Energy estimates:

$$\Gamma = \frac{1}{2}\left(\|e^z\|^2 + \|e^\theta\|^2\right)$$

$$\dot{\Gamma} = -\lambda\|Re^z\|^2 + \left\langle H_0(Re^z)e^\theta - H_0(R^\perp e^z)\theta^\star, e^z\right\rangle - \left\langle R^\perp e^z, H_0(z)e^\theta\right\rangle$$

Summary

1. Motivation
2. Estimator construction & results
3. Conclusion & outlook

Estimator construction

$$\begin{cases} \dot{z}^\star + H_0(z^\star)\theta^\star + C(q)z^\star = 0, \\ y = Rz^\star. \end{cases} \tag{$\mathcal{S}$}$$

$$\dot{y} = R\dot{z}^\star = -R\left[H_0(z^\star)\theta^\star + C(q)z^\star\right].$$

Estimators $z^\perp$ and θ of $R^\perp$ and $\theta^\star$:

$$\begin{cases} \bar{z} = y + z^\perp \\ \dot{z}^\perp + R^\perp\Big(H_0(\bar{z})\theta + C(q)\bar{z}\Big) = \Psi^\perp \\ \dot{\theta} = \Psi^\theta. \end{cases} \tag{1}$$

Similar works: Lepsien et al. 2025.

Sketch of proof

$$\begin{cases} \Psi^{\perp} = \mu\left[\tilde{H}_0(\theta)+C(q)\right]^{*} R\Big(\left[\tilde{H}_0(\theta^{\star})+C(q)\right]e^{z}+H_0(\bar{z})e^{\theta}\Big) \\ \Psi^{\theta} = \mu\left[H_0(\bar{z})\right]^{*} R\Big(\left[\tilde{H}_0(\theta^{\star})+C(q)\right]e^{z}+H_0(\bar{z})e^{\theta}\Big) \end{cases} \tag{2}$$

$$\begin{aligned} \dot{\Gamma} = &-\left\langle H_0(e^{z})\theta^{\star}+H_0(\bar{z})e^{\theta}, e^{z}\right\rangle \\ &-\mu\langle\left[\tilde{H}_0(\theta^{\star})+C(q)\right]e^{z}+H_0(\bar{z})e^{\theta}, RH_0(e^{z})e^{\theta}\rangle \\ &-\mu\left\|R\big(\left[\tilde{H}_0(\theta^{\star})+C(q)\right]e^{z}+H_0(\bar{z})e^{\theta}\big)\right\|^{2} \end{aligned}$$

Hypotheses

- continuity (and maybe coercivity) of the operators involved
- smallness of initial parameter error $\theta_0 - \theta^\star$
- Persistence of excitation: $\exists \tau > 0,\ \exists \varepsilon > 0,\ \forall T > 0,$

$$\int_T^{T+\tau} \left\| R\big(\big[\tilde{H}_0(\theta^\star) + C(q)\big] e^z + H_0(\bar{z}) e^\theta\big) \right\|^2 \geq \varepsilon \|e^\theta\|^2. \tag{3}$$

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What about the control?

$\exists q$, $\exists \tau_q > 0$, $\exists \varepsilon_q > 0$, $\forall T > 0$,

$$\int_T^{T+\tau} \left\| R\left(\left[\tilde{H}_0(\theta^\star) + C(q) \right] e_q^z + H_0(\bar{z}_q) e_q^\theta \right) \right\|^2 \geq \varepsilon \| e_q^\theta \|^2. \tag{4}$$

How could one optimise the control?

- maximisation of information $\left\| R\left(\left[\tilde{H}_0(\theta^\star) + C(q) \right] e^z + H_0(\bar{z}) e^\theta \right) \right\|^2$ over an acquisition period $[0, T]$?
- open- or closed-loop?

Conclusion & outlook

We have:

- a nice general framework
- an online method that should work

Now we need:

- prove convergence, despite additional noises
- to design efficient and robust controls
- to test it numerically

Outlook on future works:

- one derivative in time seems efficient, could we make use of multiple?
- what about other systems?

Thank you!

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Lepsien, Arthur, Philipp Kügler, and Alexander Schaum (2025). "On the Structural Analysis and Optimal Input Design for Joint State and Parameter Estimation". In: *IEEE Access* 13, pp. 211944–211959.