Computer-Assisted Proofs of Non-Reachability for Linear Parabolic Control Problems with Bounded Constraints

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Outline

1. Introduction & Convex Analysis
2. Control setting
3. Examples of computer-assisted proofs
4. Conclusion

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Context

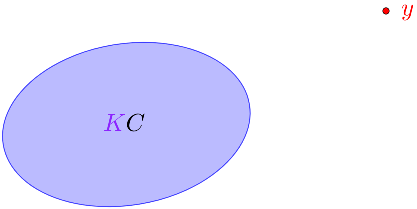
Given

- $K : X \to Y$ is a linear continuous operator, X and Y two Hilbert spaces
- $\mathrm{C} \subset X$ is a non-empty, closed, convex and bounded set
- $y \in Y$,

we address the following problem:

$$\text{Does } x \in \mathrm{C} \text{ exist such that } Kx = y\,?$$

Geometric intuition



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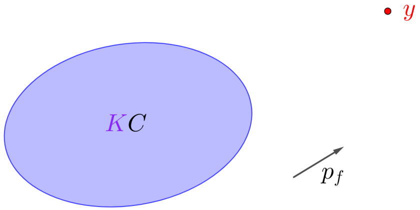

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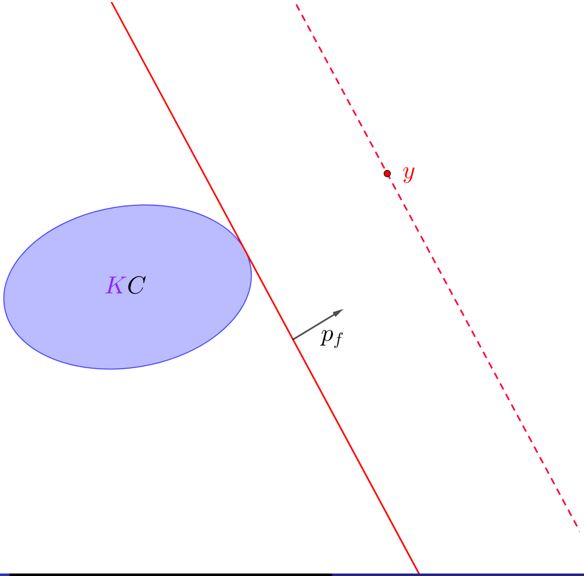

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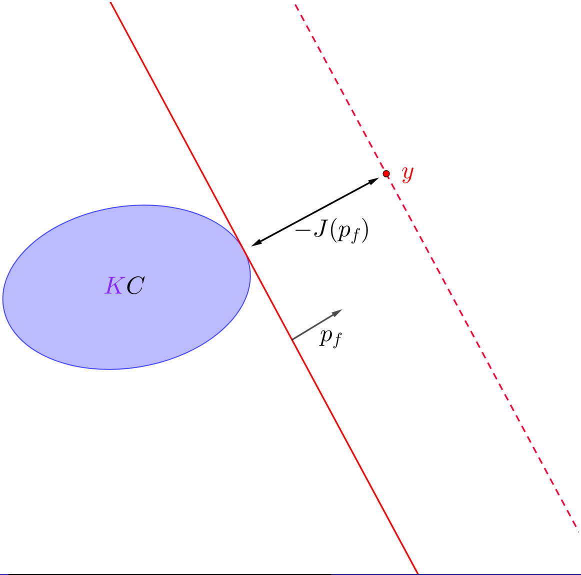

Dual functional

Denoting

$$J : p_f \mapsto \sigma_{K\mathcal{C}}(p_f) - \langle y, p_f \rangle,$$

where

$$\sigma_{K\mathcal{C}} : p_f \mapsto \sup_{z \in K\mathcal{C}} \langle p_f, z \rangle.$$

Theorem

If there exists p_f such that $J(p_f) < 0$, then there exists no $x \in \mathcal{C}$ satisfying $Kx = y$.

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$$J : p_f \mapsto \sigma_{\mathcal{C}}(K^* p_f) + \sigma_{\mathcal{Y}}(-p_f),$$

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Theorem

If there exists p_f such that $J(p_f) < 0$, then there exists no $x \in \mathcal{C}$ satisfying $Kx \in \mathcal{Y}$.

Fenchel duality

Denoting, for a set A: $$\delta_A : x \mapsto \begin{cases} 0 & \text{if } x \in A \\ +\infty & \text{if } x \notin A. \end{cases}$$

Lemma (Reformulation)

There exists $x \in \mathrm{C}$ such that $Kx = y$ if and only if

$$\inf_{x \in U} \delta_{\mathrm{C}}(x) + \delta_{\{y\}}(Kx) = 0. \qquad (\mathcal{P})$$

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Theorem (Strong duality)

$$\inf_{x \in U} \delta_{\mathrm{C}}(x) + \delta_{\{y\}}(Kx) = -\inf_{p_f \in X} J(p_f).$$

Regularisation

Consider the regularised dual problem:

$$\inf_{p_f \in X} J(p_f) + \frac{\lambda}{2}\|Ap_f\|^2.$$

Proposition

We have

$$\exists p_f \in X, \quad J(p_f) < 0 \quad \Longleftrightarrow \quad \exists p_f \in X, \quad J(p_f) + \frac{\lambda}{2}\|Ap_f\|^2 < 0$$

This allows for a large panel of primal-dual minimisation methods.

General methodology

Theorem

If there exists p_f such that $J(p_f) < 0$, then there exists no $x \in \mathcal{C}$ satisfying $Kx = y$.

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In practice, to apply this theorem, three steps are required:

1. find a proxy $J_d \simeq J$ such that we can numerically evaluate J_d
2. find p_{fd} such that $J_d(p_{fd}) < 0$
3. associate p_{fd} to some p_f and check that $J(p_f) < 0$

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   - bound discretisation errors $e_d(p_f)$
   - bound round-off errors $e_r(p_{fd})$.
   - check that $J_d(p_{fd}) + e_d(p_f) + e_r(p_{fd}) < 0$.

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Control problem

Consider the following control problem:

$$\begin{cases} \dot{y}(t) + Ay(t) = Bu(t) & \forall t \in [0, T] \\ y(0) = y_0 \in X \\ u(t) \in \mathcal{U} \subset U & \forall t \in [0, T]. \end{cases} \qquad (S)$$

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which allows the Duhamel decomposition

$$y(T, \cdot\, ; y_0, u) = S_T y_0 + L_T u.$$

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We call the constraint set

$$E_{\mathcal{U}} = \{u, \quad \forall t \in [0, T],\ u(t) \in \mathcal{U}\} \subset L^2(0, T; U),$$

where $\mathcal{U}$ will be assumed to be non-empty, closed, convex and bounded in U by $M > 0$.

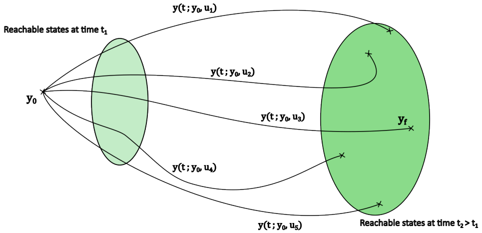
Reachability

Definition

A target y_f is $\mathcal{U}$-reachable from y_0 in time T if :

$$\exists\, u \in E_{\mathcal{U}}, \quad y(T, \cdot\,; u) = y_f.$$

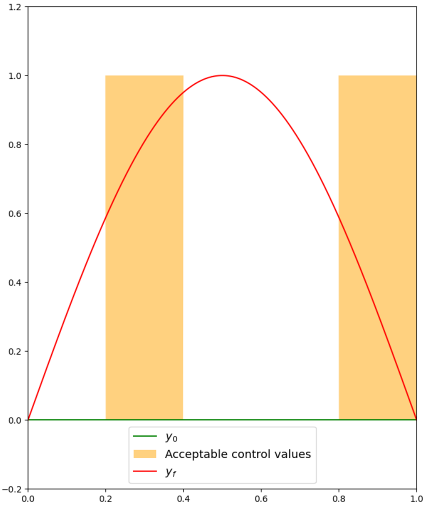
The reachable set $S_T y_0 + L_T E_{\mathcal{U}}$ is the set of all $\mathcal{U}$-reachable points (from y_0 in time T).



Control of the 1D heat equation

$\forall\, t, x \in [0, T] \times [0, 1],$

$$\begin{cases} \dot{y}(t,x) - \Delta y(t,x) = \mathbb{1}_\omega u(t,x) \\ y(0,x) = y_0(x) = 0 \\ y(t,0) = y(t,1) = 0 \\ 0 \leq u(t,x) \leq 1 \\ y(T,x) = y_f(x) = \sin(\pi x). \end{cases}$$



State of the Art

Control under constraints

- Symmetrical constraints: Berrahmoune (2020), Chen & Rosier (2022)
- Unilateral constraints: Pighin & Zuazua (2018), Antil & al. (2024), Pouchol, Trélat & Zhang (2024)
- Bounded constraints: Wang (2008), Casas & Kunish (2022)

Link to convex analysis

Theorem

The three following assertions are equivalent:

- y_f *is not* $\mathcal{U}$*-reachable from* y_0 *in time* T
- $\tilde{y}_f := y_f - S_T y_0 \notin L_T E_{\mathcal{U}}$
- $\exists p_f \in X, \quad \sigma_{E_{\mathcal{U}}}(L_T^* p_f) - \langle \tilde{y}_f, p_f \rangle < 0$

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And:

$$L_T^* : \begin{cases} X \to U \\ p_f \mapsto (t \mapsto B^* p(t)), \end{cases}$$

where $t \mapsto p(t)$ solves the adjoint equation

$$\begin{cases} \dot{p}(t) = A^* p(t), \\ p(T) = p_f. \end{cases} \tag{$\mathcal{A}$}$$

Hypotheses

Suppose that:

- $V \subset X$ are Hilbert spaces, V dense and continuously embedded in X.
- $A : \mathcal{D}(A) \subset V \to X$, such that A^* is continuous and coercive, that is $\exists 0 < a_0 \leq a_1$ satisfying

$$\forall v, w \in \mathcal{D}(A^*) \times V, \quad \begin{cases} |\langle A^* v, w \rangle| \leq a_1 \|v\|_V \|w\|_V \\ \mathrm{Re}(\langle A^* v, v \rangle) \geq a_0 \|v\|_V^2. \end{cases}$$

- $B : U \to X$ is bounded.

Discretisation

Let $h > 0$ and a finite-dimensional subset $V_h \subset V$ such that

$$\forall f \in X, \quad \inf_{v_h \in V_h} \|A^{-1} f - v_h\|_V + \inf_{v_h \in V_h} \|(A^*)^{-1} f - v_h\|_V \leq C_0\, h \|f\|,$$

We consider a space-discretisation over V_h and a implicit Euler time-discretisation of $(\mathcal{A})$ with time step Δt and get the following result:

Proposition

$\forall (p_f, p_{fh}) \in X \times V_h, \quad \forall n \in \{0, \ldots, N_t\},$

$$\|p(t_n) - p_{h,n}\| \leq C_1 \|p_f - p_{fh}\| + \left(C_2 h^2 + C_3 \Delta t\right) \|A^* p_f\|,$$

where C_1, C_2 and C_3 are known explicitly and depend only on a_0 and a_1.

Error control

Discretisation errors

Consider

$$J_{\Delta t,h}(p_{fh}) = \Delta t \sum_{n=1}^{N_t} \sigma_{\mathcal{U}}\left(B^*\left(\mathrm{Id} - \Delta t A_h^*\right)^{-n} p_{fh}\right) - \langle y_f, p_{fh}\rangle + \left\langle y_0, \left(\mathrm{Id} - \Delta t A_h^*\right)^{-N_t} p_{fh}\right\rangle .$$

Assume furthermore that for $p_{fh} \in V_h$, you know how to compute explicit $\sigma_{\mathcal{U}}(B^* p_{fh})$, $\langle y_f, p_{fh}\rangle$ and $\langle y_0, p_{fh}\rangle$.

Theorem

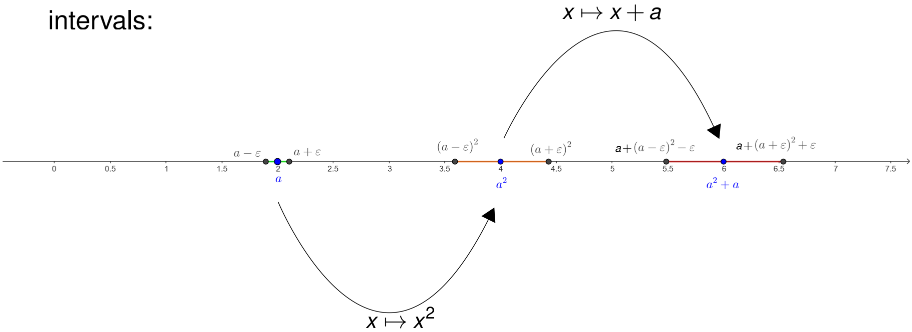
For all $p_f \in \mathcal{D}(A^*)$, $p_{fh} \in V_h$, we then have

$$\begin{aligned} |J(p_f) - J_{\Delta t,h}(p_{fh})| \leq & \tfrac{1}{2} MT\|B\| \, \Delta t \|A^* p_f\| \\ & + (\|y_0\| + MT\|B\|)\left(C_2 h^2 + C_3 \, \Delta t\right)\|A^* p_f\| \\ & + \left((\|y_0\| + MT\|B\|)\, C_1 + \|y_f\|\right)\|p_f - p_{fh}\|. \end{aligned}$$

Error control

Round-off errors

To take into account round-off errors made by during computations on finite-byte machines, one has to propagate all potential errors using intervals:



The Intlab library, encoded in Matlab by Siegfried M. Rump, takes care of it for us.

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General methodology

Theorem

If there exists p_f such that $J(p_f) < 0$, then y_f is not $\mathcal{U}$-reachable for (S) in time T.

In practice, to apply this theorem, three steps are required:

1. discretise J into $J_{\Delta t,h} \simeq J$ such that we can evaluate $J_{\Delta t,h}$
2. find p_{fh} such that $J_{\Delta t,h}(p_{fh}) < 0$
3. associate p_{fh} to some p_f and check that $J(p_f) < 0$:
   - if needed, interpolate p_{fh} into p_f
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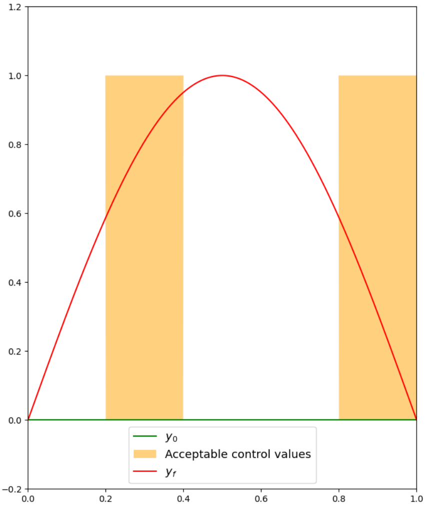

Choice of V_h

Here we have:

- $X = L^2(0,1)$ the state space
- $V = H_0^1(0,1)$ and $\mathcal{D}(A) = \mathcal{D}(A^*) = H_0^1(0,1) \cap H^2(0,1)$.

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Two choices of space discretisation are possible:

1. $V_h \subset \mathcal{D}(A)$ (cubic splines, spectral methods...):
    - Pros: no interpolation needed, $p_f = p_{fh} \implies \|p_{fh} - p_f\| = 0$
    - Cons: closed formulas more complicated (when possible), heavy computation costs

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2. $V_h \not\subset \mathcal{D}(A)$ ($\mathbb{P}_1$ finite elements, ...):
    - Pros: easier computations and many closed formulas
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Control of the heat equation

$\forall t, x \in [0, T] \times [0, 1]$

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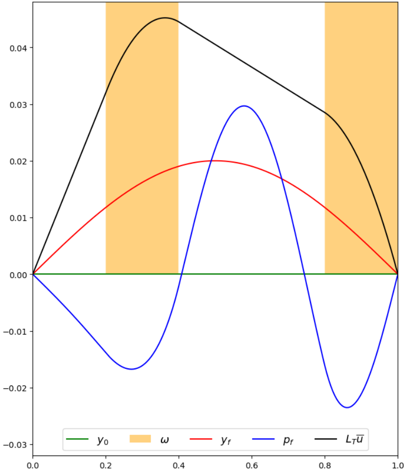
Proposition

y_f is not $\mathcal{U}$-reachable from y_0 in time $T = 1$. Indeed,

$$J(p_f) \in [-0.0093, -0.0035] < 0.$$



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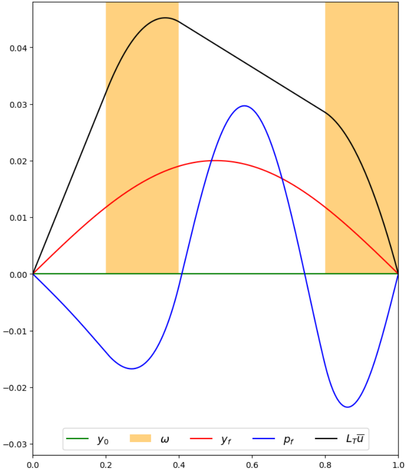
Proposition

The minimal time $t^\star$ required to steer y_0 to y_f satisfies:

$$t^\star \geq 1.15.$$

Indeed,

$$J(p_f; 1.15) \in [-0.0073, -4 \cdot 10^{-5}] < 0.$$



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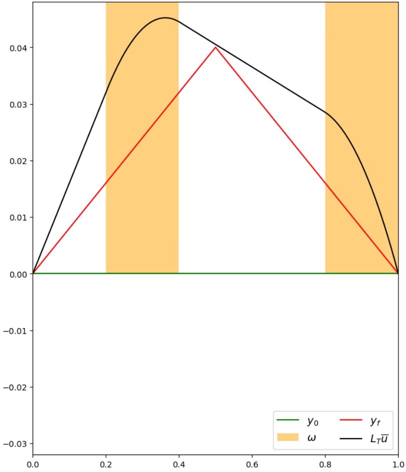

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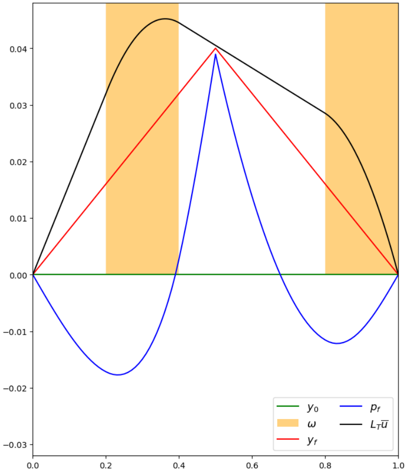

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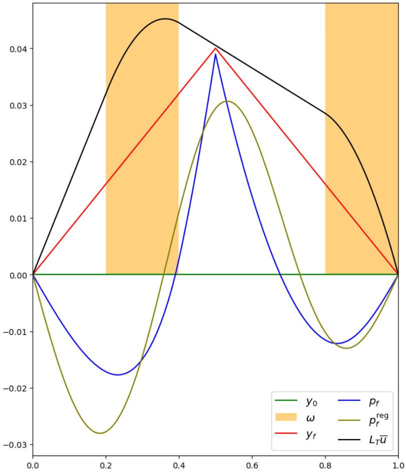
Proposition

y_f is not $\mathcal{U}$-reachable from y_0 in time $T = 1$. Indeed,

$$J(p_f^{reg}) \in [-0.0049, -6 \cdot 10^{-5}] < 0.$$



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Conclusion & Perspectives

Contributions :

- A general method to analyse the non-reachability of targets of linear control problems
- Fine explicit estimates for a wide class of parabolic control problems

Perspectives :

- Apply the method for other classes of linear PDEs
- For ODEs, develop a method to prove numerically the reachability of a given target and approximate the reachable set with guaranteed sets

Thank you for you attention!