Computer-assisted proofs of non-reachability for linear parabolic control systems

Ivan Hasenohr

PhD student under the supervision of Camille Pouchol, Yannick Privat et Christophe Zhang

Université Paris Cité

Congrès des Jeunes Chercheur.e.s en Mathématiques Appliquées

Outline

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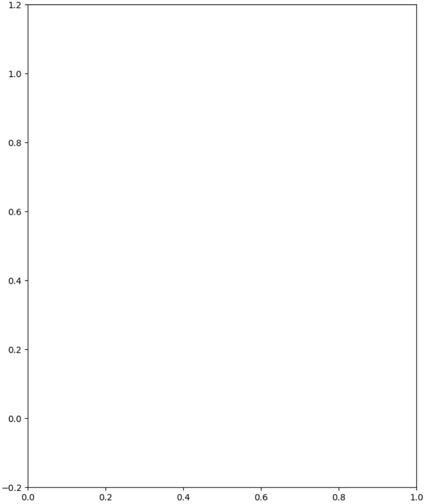
Motivation

$$\forall\; t, x \in [0, T] \times [0, 1],$$

$$\begin{cases} \dot{y}(t,x) = \Delta y(t,x) \\ y(0,x) = 0 \\ y(t,0) = y(t,1) = 0. \end{cases}$$



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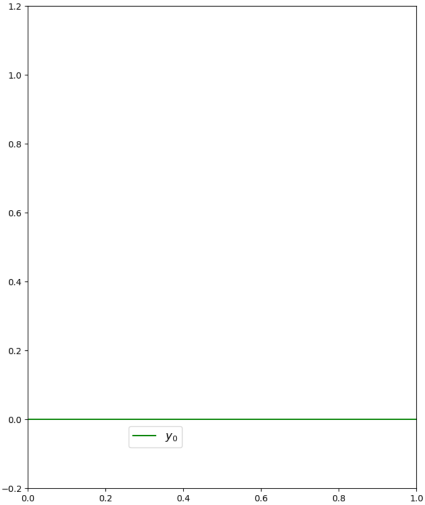

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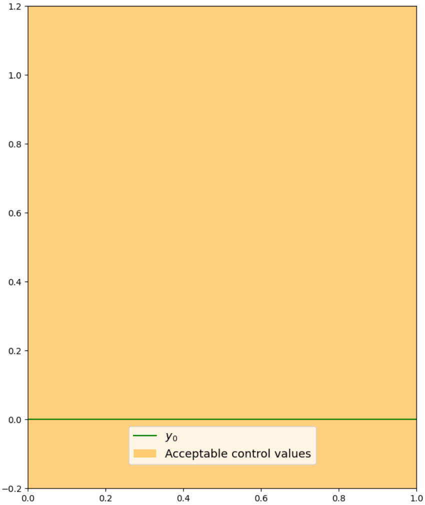

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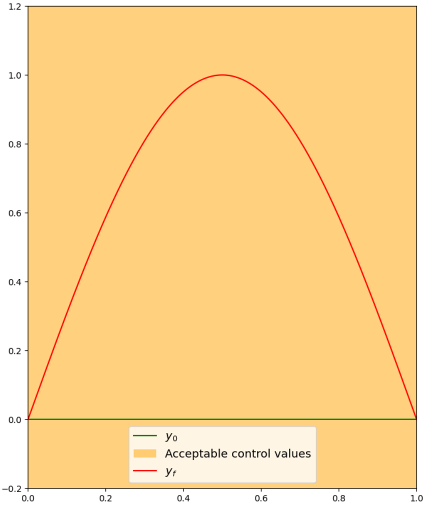

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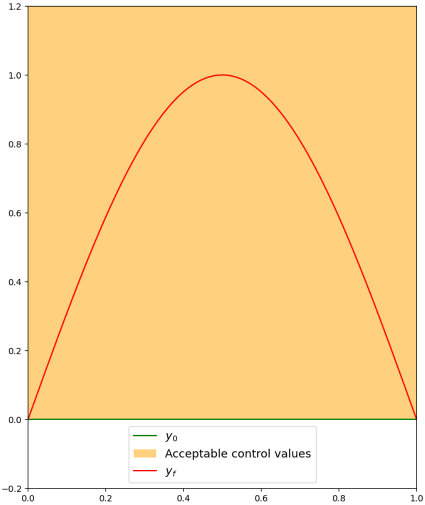

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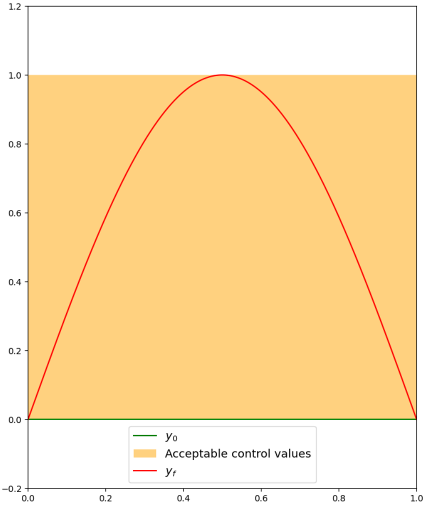

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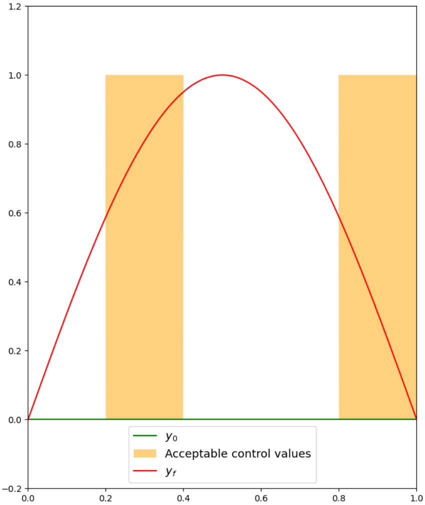

Control system

We call linear control system

$$\begin{cases} \dot{y}(t) = Ay(t) + Bu(t) & \forall t \in [0,T] \\ y(0) = y_0 & \\ u(t) \in \mathcal{U}_0 & \forall t \in [0,T]. \end{cases} \qquad (S)$$

Which allows the Duhamel decomposition

$$y(T,\cdot\,; y_0, u) = S_T y_0 + L_T u.$$

We call the constraint set

$$\mathcal{U} = \left\{ u, \quad \forall t \in [0,T],\ u(t) \in \mathcal{U}_0 \right\}.$$

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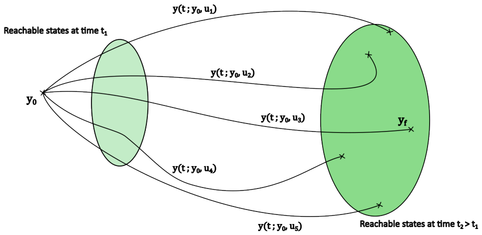
Reachability

Definition

A target y_f is $\mathcal{U}$-reachable in time T if :

$$\exists\, u \in \mathcal{U}, \quad y(T, \cdot\, ; u) = y_f.$$

We call reachable set and denote $L_T\mathcal{U}$ the set of all $\mathcal{U}$-reachable points.



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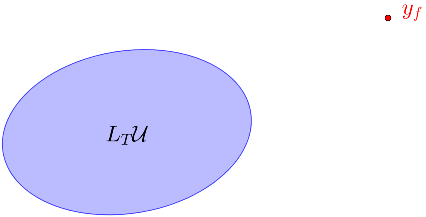

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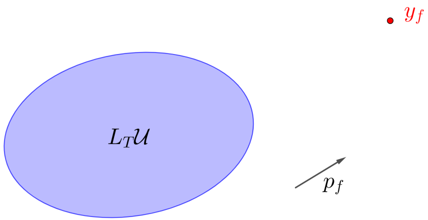

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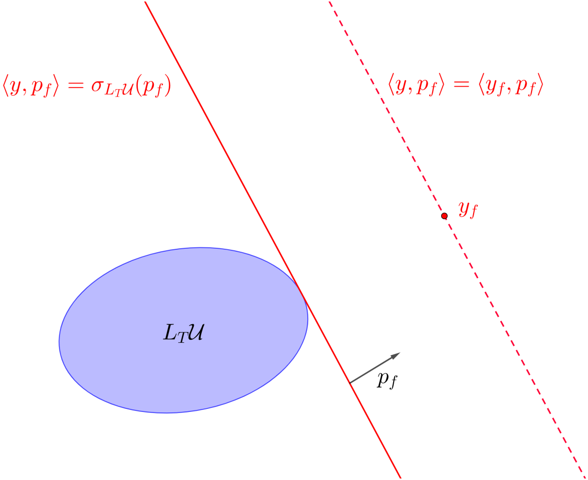

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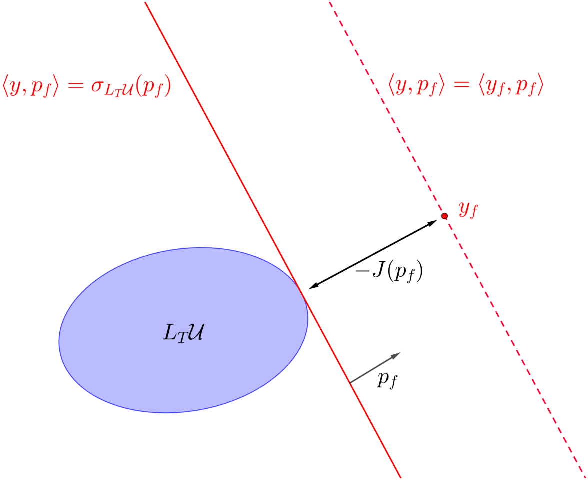

Non-reachability : theorem

Denoting

$$J : p_f \mapsto \sigma_{L_T \mathcal{U}}(p_f) - \langle p_f, y_f \rangle,$$

where

$$\sigma_{L_T \mathcal{U}} : p_f \mapsto \sup_{x \in L_T \mathcal{U}} \langle x, p_f \rangle.$$

Theorem

If there exists p_f such that $J(p_f) < 0$, then y_f is not $\mathcal{U}$-reachable for (S) in time T.

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To use this theorem to prove the non-reachability of y_f, three steps are required :

1. discretise J into $J_{\Delta t,h} \simeq J$ such that we can evaluate $J_{\Delta t,h}$
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Computing J

J can be reformulated as :

$$J : p_f \mapsto \int_0^T \sigma_{\mathcal{U}_0}\left(B^* S_{T-t}^* p_f\right) \mathrm{d}t - \langle p_f, y_f \rangle,$$

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Time discretisation – Implicit Euler scheme :

$$J_{\Delta t}(p_f) = \Delta t \sum_{n=0}^{N_t-1} \sigma_{\mathcal{U}_0}\left(B^* (\mathrm{Id} - \Delta t A^*)^{-(Nt-n)} p_f\right) - \langle p_f, y_f \rangle,$$

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Space discretisation – P1 finite elements :

$$J_{\Delta t, h}(p_{fh}) = \Delta t \sum_{n=0}^{N_t-1} \sigma_{\mathcal{U}_0}\left(B_h^* (\mathrm{Id} - \Delta t A_h^*)^{-(Nt-n)} p_{fh}\right) - \langle p_{fh}, y_{fh} \rangle.$$

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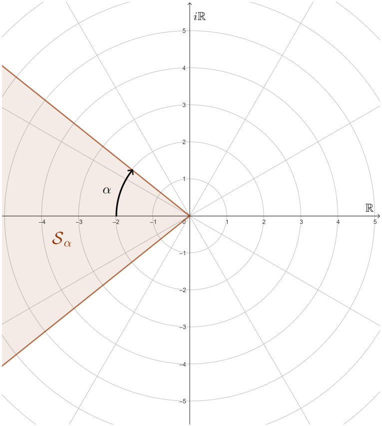

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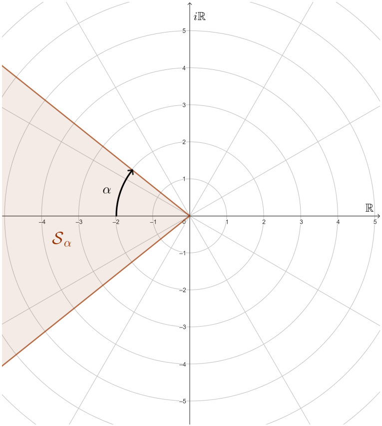

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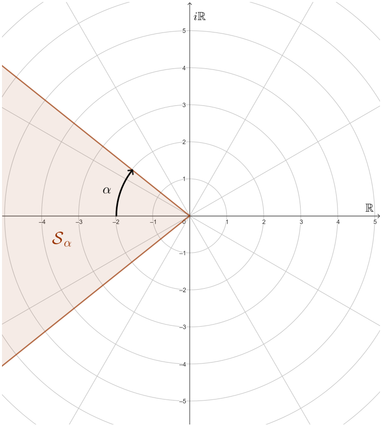

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- Continuity : for $0 < a_0 \leq a_1$,

$$\forall v, w \in \mathcal{D}(A) \times V, \quad |\langle Av, w\rangle| \leq a_0 \|v\| \|w\|.$$

Discretisation error

Theorem

Let $p_{fh} \in V_h$ and $p_f \in \mathcal{D}(A^*)$ such that $\forall i \in \{0,\dots,N_x\}, p_{fh}(ih) = p_f(ih)$. Then, if A satisfy the previously stated hypotheses, we have

$$e_d(p_f) = |J(p_f) - J_{\Delta t,h}(p_{fh})| \leq \left(C_1\Delta t + C_2 h^2\right)\|A^* p_f\|$$

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with

$$C_1 = \frac{1}{2} MT\|B\| + \frac{C_\alpha}{\cos(\alpha)} \Big(\|y_0\|_X + MT\|B\|\Big)$$

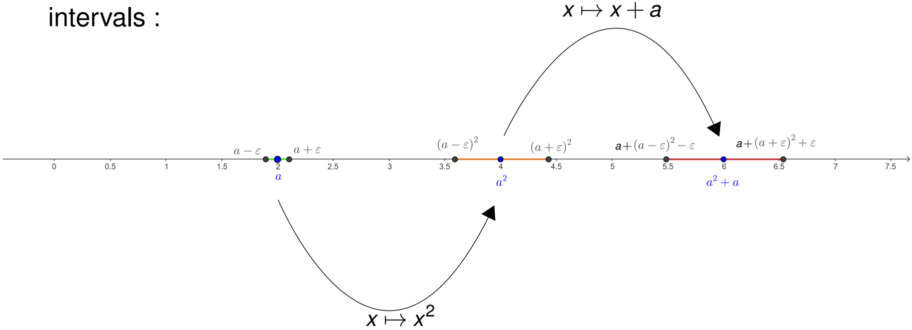
$$C_2 = \left(\frac{a_1^2 C_0^2}{a_0} \left(7 + \frac{4\ln(2)}{\cos(\alpha)} + C_\alpha\right) + C_\alpha C_0\right) \Big(\|y_0\|_X + MT\|B\|\Big)$$

$$C_\alpha \leq 2 + \frac{2}{\sqrt{3}} \qquad \& \qquad C_0 = \frac{1}{2}.$$

Round-off errors

Interval arithmetic

To take into account round-off errors made by during computations on finite-byte machines, one has to propagate all potential errors using intervals :



The Intlab library, encoded in Matlab by Siegfried M. Rump, takes care of it for us.

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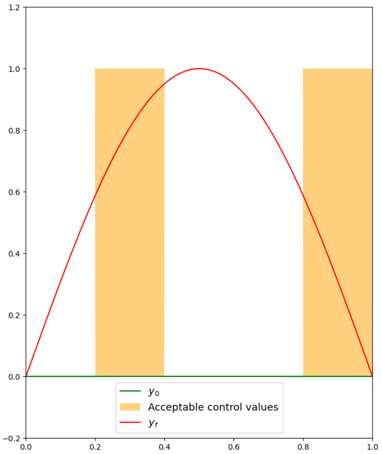
Numerical example

Back to the original example :
$\forall t, x \in [0, T] \times [0, 1]$

$$\begin{cases} \dot{y}(t,x) = \Delta y(t,x) + \mathbb{1}_\omega u(t,x) \\ y(0,x) = y_0(x) = 0 \\ y(t,0) = y(t,1) = 0 \\ 0 \leq u(t,x) \leq 1 \\ y(T,x) = y_f(x) = \sin(\pi x). \end{cases}$$



Numerical example

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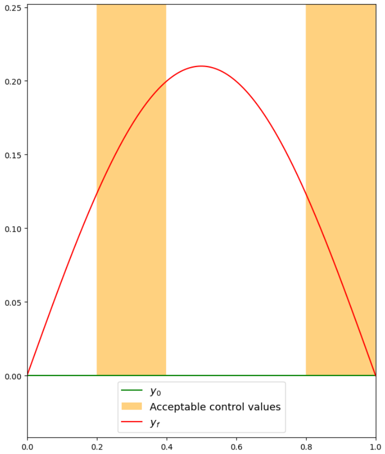
$\forall t, x \in [0, T] \times [0, 1]$

$$\begin{cases} \dot{y}(t,x) = 0.1\Delta y(t,x) + \mathbb{1}_\omega u(t,x) \\ y(0,x) = 0 \\ y(t,0) = y(t,1) = 0 \\ 0 \leq u(t,x) \leq 1 \\ y(T,x) = y_f(x) = 0.21\sin(\pi x). \end{cases}$$



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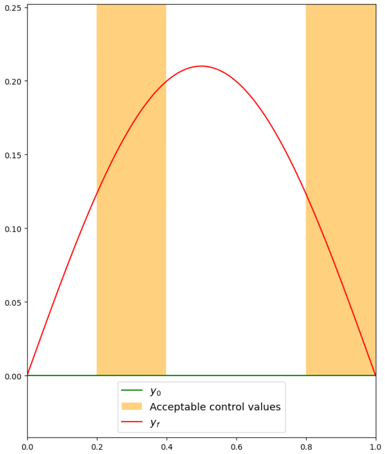
$$y_f = 0.21\sin(\pi\cdot).$$

Is y_f $\mathcal{U}$-reachable in time $T = 1$?



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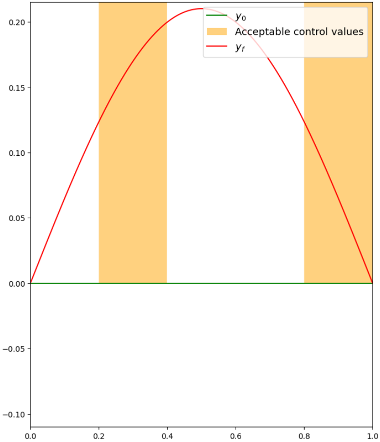

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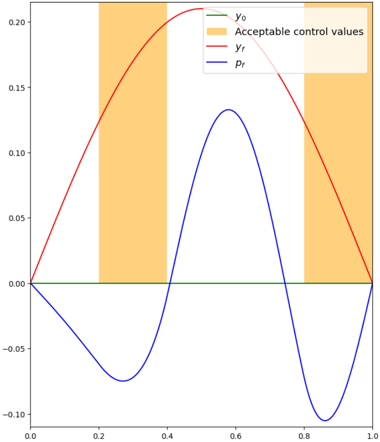
Proposition

y_f is not $\mathcal{U}$-reachable in time $T = 1$. Indeed,

$$J(p_f) \in [-0.0093, -0.0035] < 0.$$



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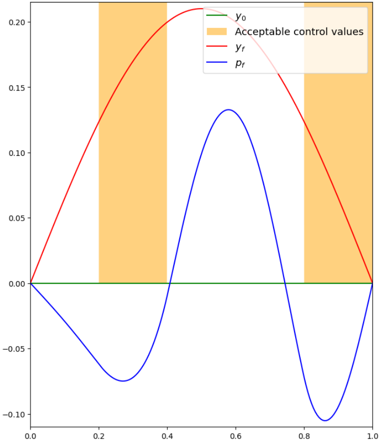

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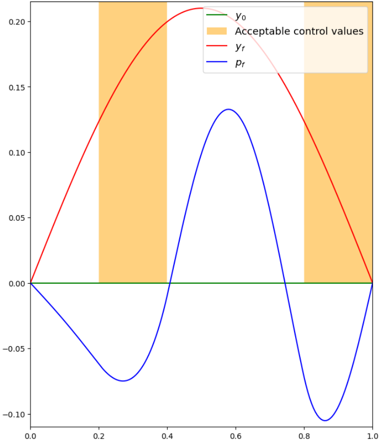
$$t^\star \geq 1.1582.$$

Indeed,

$$J(p_f; t^\star) \in [-0.0073, -4 \cdot 10^{-5}] < 0.$$



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Cubic spline interpolation

For a given p_{fh} P1 element, we need to find $p_f \in \mathcal{D}(A)$ such that

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Lemma

If $A = \Delta$, cubic splines are the optimal way to interpolate p_{fh}, in the sense that

$$\inf_{p_f \in \mathcal{D}(\Delta)} \|\Delta p_f\| = \inf_{\substack{p_f \in \mathcal{D}(\Delta) \\ p_f \text{ cubic spline}}} \|\Delta p_f\|.$$

Furthermore, the $\inf$ is reached, the optimal spline has a closed-form expression satistying $p_f \in \mathcal{C}^2([0,1])$.

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Furthermore, the inf *is reached, the optimal spline has a closed-form expression satisfying $p_f \in \mathcal{C}^2([0,1])$.*

Recall :

$$e_d(p_f) = |J(p_f) - J_{\Delta t,h}(p_{fh})| \leq \left(C_1 \Delta t + C_2 h^2\right) \|\Delta p_f\|.$$

Outline

Conclusion and perspectives

Contributions :

- A general method to analyse the non-reachability of targets of linear control problems
- Fine explicit estimates for a wide class of parabolic control problems

Perspectives :

- Apply the method for other classes of linear PDEs
- For ODEs, develop a method to prove numerically the reachability of a given target and approximate the reachable set with guaranteed sets (*work in progress*)