# Rigorous approximations of reachable sets

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## 1. Context

We consider the finite dimensional linear control system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) & \forall t \in [0,T] \\ x(0) = 0 \\ u(t) \in \mathcal{U}_0 & \forall t \in [0,T] \end{cases}$$

where $A : X \to X$ and $B : Y \to X$ are two linear continuous operators, $\mathcal{U}_0$ is a compact convex set of $Y$ and $X$ and $Y$ are two finite dimensional Hilbert spaces. $\mathcal{U} \subset L^2(0,T;Y)$ is the set of controls that verifies the constraints $\mathcal{U}_0$ at all times. Furthermore, we denote $L_T$ the linear map such that

$$\forall u \in \mathcal{U}, \quad x(T;u) = L_T u.$$

$L_T\mathcal{U}$ is therefore the reachable set of the control system. We will call support function of $\mathcal{U}$

$$\sigma_{\mathcal{U}} : \begin{cases} L^2(0,T;Y) & \to \mathbb{R} \\ v & \mapsto \sup_{u \in \mathcal{U}} \langle u, v \rangle. \end{cases}$$

## 2. Objectives

In this context, the goals are twofold:

- given a target, constructing a method to numerically prove if a target is not reachable: given $y_f$, does one have $y_f \in L_T\mathcal{U}$ ?
- approximate the reachable set with two polytopes, the first one containing targets proved to be reachable, the second one excluding points proved not to be reachable.

## 3. Computer-assisted proofs

Two kinds of errors need to be bounded in order to obtain a mathematically rigorous computer-assisted proof:

- Discretisation errors $e_d$ due to time-discretisation with time step $\Delta t$
- Rounding-off errors $e_r$

The total error bound between $J$ and $J_{\Delta t}$ - the discretised functional - hence is

$$\forall p_f \in X^*, \quad |J(p_f) - J_{\Delta t}(p_f)| \leq e_d(p_f) + e_r(p_f).$$

### 1 Discretisation errors

Let us consider $(p_n)_{n \in \{0,\dots,N_t\}}$ the discretisation of the following differential equation using the Euler implicit scheme:

$$\begin{cases} \dot{p}(t) = A^* p(t) & \forall t \in [0,T] \\ p(0) = p_f, \end{cases}$$

Suppose $-A^*$ to be m$\alpha$-accretive, where $\alpha \leq \frac{\pi}{2}$. Then $\forall n \in \{0,\dots,N_t\}$,

$$\|p(n\Delta t) - p_n\| \leq \Delta t \ \min\left(\frac{1+\sqrt{2}}{\cos(\alpha)}\|A^* p_f\|, \frac{1}{2} t_n \|(A^*)^2 p_f\|\right).$$

Finally, $\forall p_f \in X^*$:

$$e_d(p_f) \leq \Delta t \ M\|B\| \left(\frac{1}{2}T\|A\|\|p_f\| + \sum_{n=0}^{N_t-1} \|p(n\Delta t) - p_n\|\right) + \|y_0\|\|p(T) - p_{N_t}\|,$$

where $M > 0$ is such that $\mathcal{U}_0 \ \subset \ \mathcal{B}(0,M)$.

### 2 Round-off errors

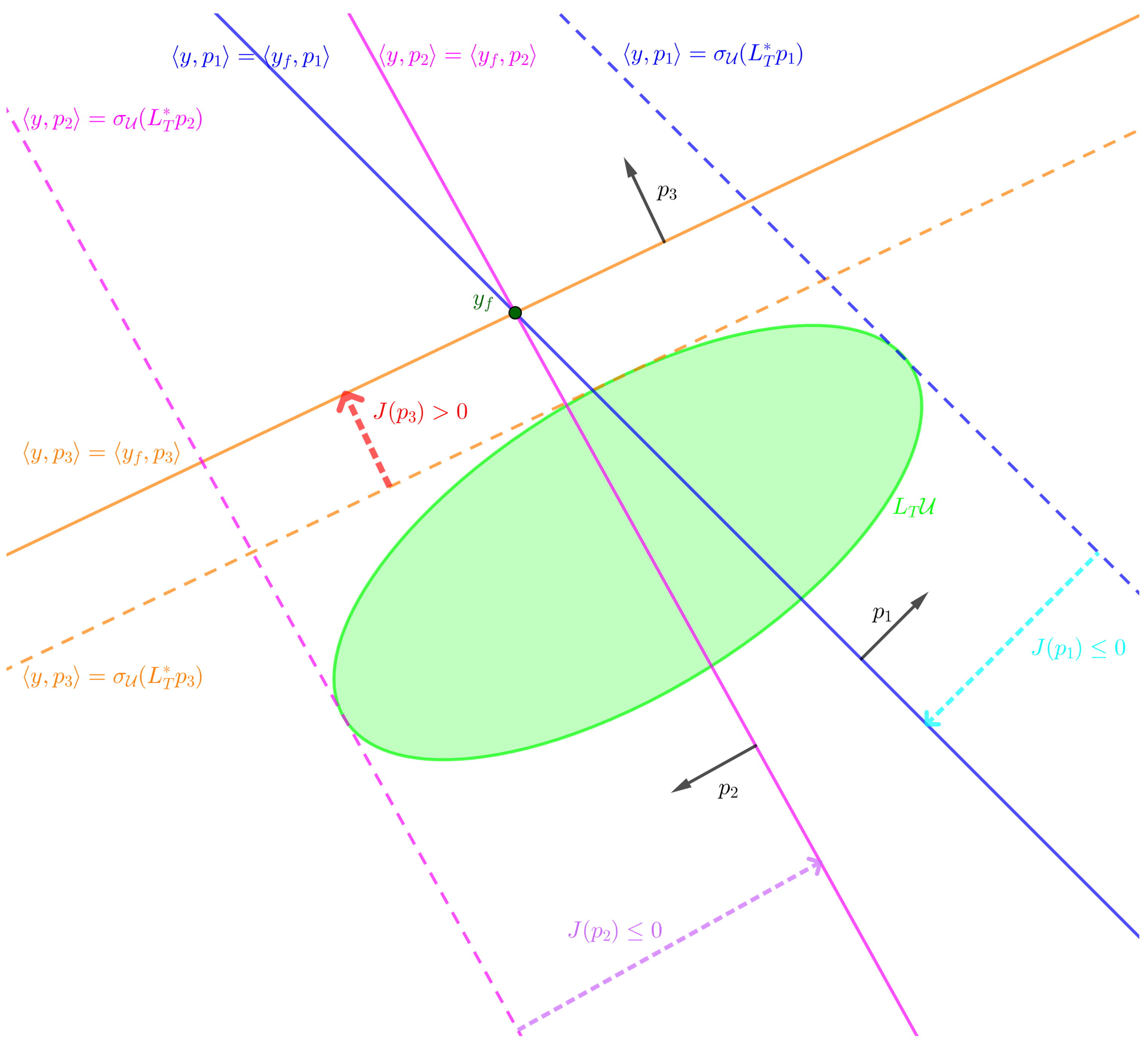
Round-off errors are bounded at each step using interval arithmetic: each variable is ascribed to an interval into which it is guaranteed to belong, and computations are done to these intervals directly. Numerically, it is computed using the INTLAB toolbox [2].

## 4. Non-reachability certificates

Considering $x_f \in X$, we have the following implication:

$$\exists p_f \in X^*, \quad J(p_f) := \langle p_f, x_f \rangle - \sigma_{\mathcal{U}}(L_T^* p_f) > 0 \implies x_f \notin L_T\mathcal{U}.$$

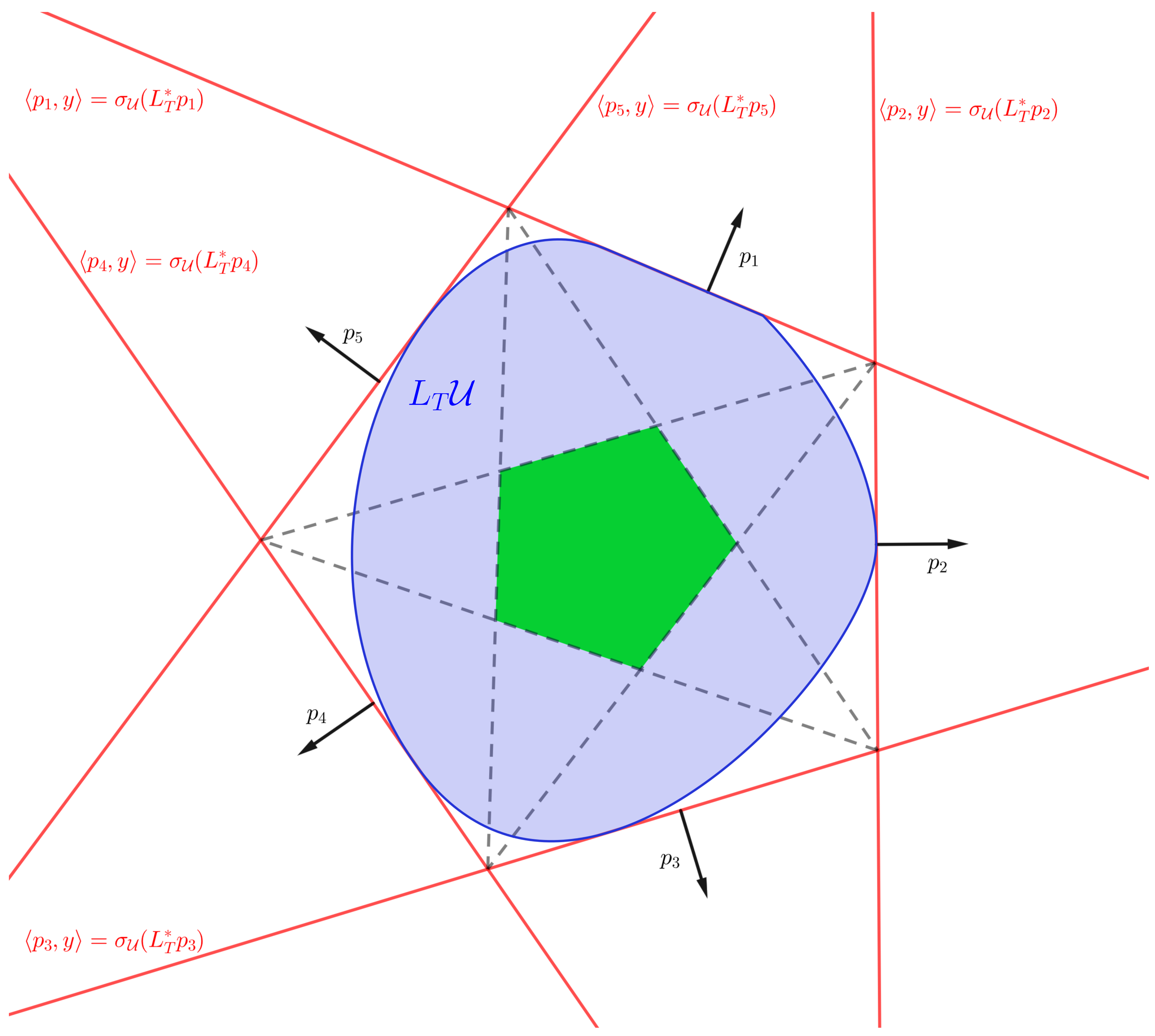
- **Computed-assisted proofs**: finding a suitable $p_f$ where a discretised functional is positive, along with error estimates, enables numerical non-reachability proofs
- **Duality algorithms**: exploiting the underlying duality, primal-dual algorithms ([1]) allow for efficient computations
- **Infinite-dimensional case**: this method might be extendable to partial differential equations, while considering space discretisation added error terms




## 5. Reachable and non-reachable polytopes

If we compute enough support hyperplanes to the reachable set, it is possible to compute two polytopes:

- the first one is described by the support hyperplanes and is guaranteed to contain the reachable set
- the second one can be computed from the support hyperplanes (in 2D, as shown in the following figure), and is guaranteed to be contained in the reachable set.




## References

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[2] S.M. Rump. "INTLAB - INTerval LABoratory". In: *Developments in Reliable Computing*. Ed. by Tibor Csendes. `http://www.tuhh.de/ti3/rump/`. Dordrecht: Kluwer Academic Publishers, 1999, pp. 77–104.
[3] R. Baier et al. "Approximation of reachable sets by direct solution methods for optimal control problems". In: *Optimization Methods and Software* 22.3 (2007), pp. 433–452. DOI: `10.1080/10556780600604999`.
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